Chapter 1. Magnetic Circuit

1.1 Magnetic Circuit (Lecture 1)

Consider a simple magnetic structure as shown in Figure 1.1. The following assumptions are made for simplifying magnetic circuit analysis:

(A1) The magnetic flux is restricted to flow through the magnetic materials with no leakage;

(A2) The magnetic flux density is uniform within the magnetic materials.

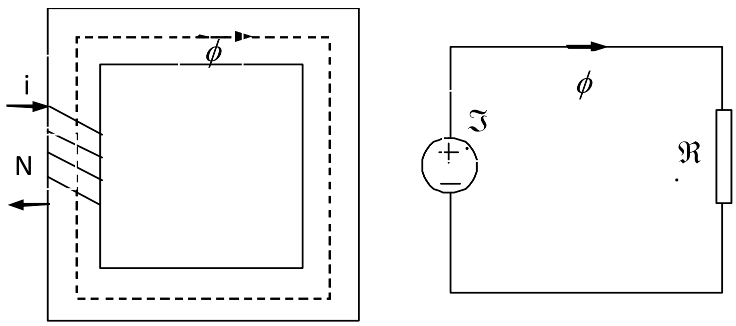

Figure 1.1

If the coil has N turns and carries a current i, the magnetomotive force (mmf) in $A \cdot t$, produced by the current i, is

$$\mathcal{F} = Ni$$

Similar to the voltage in an electric circuit, the magnetovoltage force has polarities, which can be determined by the right hand rule: If the coil is grasped in the right hand with the fingers pointing in the direction of the current, the thumb will point to the positive palarity of the mmf.

Ampere's law states that the line integral of the tangential component of the magnetic field intensity $\vec{H}$ (in A/m) around a closed path C is equal to the total current passing through the surface enclosed by the path, that is,

$$\mathcal{F} = \oint \vec{H} \cdot d\vec{l} = \oint H \cos\theta dl$$

where $\vec{l}$ is the length vector whose direction is chosen in a way so that the angle between $\vec{H}$ and $d\vec{l}$ is the smallest and θ is the angle between the vectors $\vec{H}$ and $d\vec{l}$. The direction of $\vec{H}$ is determined by the right-hand rule:

The right-hand rule 1: Imagine a current-carrying conductor held in the right hand with the thumb pointing in the direction of current flow, the fingers then point in the direction of the magnetic field created by that current.

The right-hand rule 2: If the coil is grasped in the right hand with the fingers pointing in the direction of the current, the thumb will point in the direction of the magnetic field.

Due to (A1), the mean path can be chosen to calculate the magnetic field intensity $\vec{H}$. Note that $\theta = 0$. Thus,

$$\mathcal{F} = Hl$$

where l is the mean length of the magnetic core.

The magnetic field intensity $\vec{H}$ is related to the magnetic flux density $\vec{B}$ (in Wb/m^2) by

$$\vec{B} = \mu\vec{H}$$

where $\mu = \mu_r\mu_0$ is called the magnetic permeability (in H/m) with μ_r the relative permeability and $\mu_0 = 4\pi \times 10^{-7} H/m$ the permeability of the air or free space.

The flux in the core is determined by

$$\phi = BA$$

where A represents the cross-sectional area of the magnetic core.

Therefore, $\mathcal{F} = Hl$ can be rewritten as

$$\mathcal{F} = Hl = \frac{B}{\mu}l = \frac{\frac{\phi}{A}}{\mu}l = \frac{l}{\mu A}\phi = \Re\phi$$

where $\Re = \frac{l}{\mu A}$ is defined as the reluctance of the magnetic circuit.

Comparing the expression $\mathcal{F} = \Re\phi$ with Ohm's law $V = RI$, we find that $\Re$ is analogous to R, ϕ to I, and $\mathcal{F}$ to V. This analogy enables us to represent the magnetic core in terms of an equivalent magnetic circuit as shown in Figure 1.1. Like the voltage source in the electric circuit, the mmf in the magnetic circuit has a polarity. The positive end of the mmf source is the end from which the flux exits and the negative end is the end at which the flux re-enters.

Reluctances in a magnetic circuit obey the same rules as resistances in an electric circuit. The equivalent reluctance of a number of reluctances in series is just the sum of the individual reluctances:

$$\Re_{eq} = \Re_1 + \Re_2 + \cdots$$

Similarly, reluctances in parallel combine according to the equation

$$\frac{1}{\Re_{eq}} = \frac{1}{\Re_1} + \frac{1}{\Re_2} + \cdots$$

Example 1.1:

A magnetic core is shown in Figure 1.2. Both depth and width are 3cm and its mean length is 30cm. The length of the air-gap is 0.05cm. The coil has 500 turns. The relative permeability of the core is assumed to be 70,000. Neglect fringing effects and assume the flux density of the core is $B_c = 1.0 Wb/m^2$. Find the reluctances of the core and air-gap, flux in the core, and the current required.

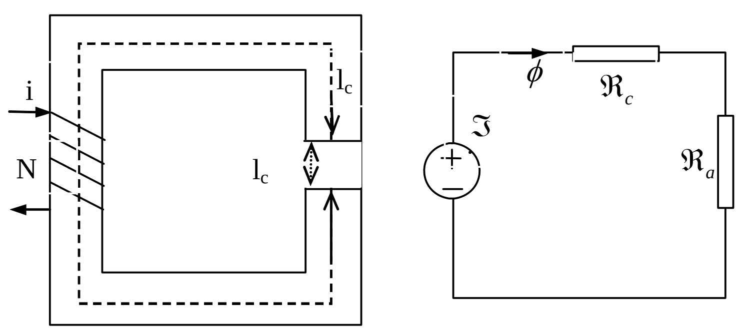

Figure 1.2 Magnetic circuit

Solution: The reluctance of the core is calculated by

$$\Re_c = \frac{l_c}{\mu_r\mu_0 A_c} = \frac{30\times10^{-2}}{70000\times4\pi\times10^{-7}\times0.03\times0.03} = 3789.4 A\cdot t/Wb$$

The reluctance of the air-gap is

$$\Re_a = \frac{l_a}{\mu_0 A_a} = \frac{0.05\times10^{-2}}{4\pi\times10^{-7}\times0.03\times0.03} = 442100 A\cdot t/Wb$$

The total reluctance in the magnetic circuit is given by

$$\Re = \Re_c + \Re_a = 3789.4 + 442100 = 445890 A\cdot t/Wb$$

The flux in the magnetic circuit is

$$\phi = B_c A_c = 1.0\times0.03\times0.03 = 0.0009 Wb$$

The current in the coil is

$$i = \frac{\phi\Re}{N} = \frac{0.0009\times445890}{500} = 0.8026 A$$

Example 1.2: Consider the magnetic circuit as shown in Figure 1.3. Determine the flux through various magnetic paths.

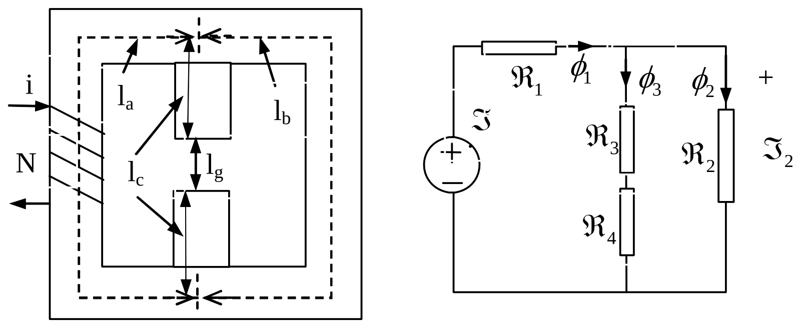

Figure 1.3 Magnetic circuit

Solution: The reluctance in the center leg is $\Re_{center} = \Re_3 + \Re_4$

The total reluctance seen from the coil side is

$$\Re_{total} = \Re_1 + (\Re_3+\Re_4)\|\Re_2 = \Re1 + \frac{(\Re_3+\Re_4)\Re_2}{\Re_3+\Re_4+\Re_2} = \frac{\Re_1(\Re_3+\Re_4+\Re_2)+(\Re_3+\Re_4)\Re_2}{\Re_3+\Re_4+\Re_2}$$

The flux in the left leg is

$$\phi_1 = \frac{\mathcal{F}}{\Re_{total}} = \frac{Ni}{\frac{\Re_1(\Re_3+\Re_4+\Re_2)+(\Re_3+\Re_4)\Re_2}{\Re_3+\Re_4+\Re_2}}$$

The flux in the center leg is

$$\phi_3 = \frac{\mathcal{F}_2}{\Re_{center}} = \frac{\phi_1(\Re_{center}\|\Re_2)}{\Re_{center}} = \frac{\phi_1}{\Re_{center}}\frac{\Re_{center}\Re_2}{\Re_{center}+\Re2} = \frac{\phi_1\Re_2}{\Re_3+\Re_4+\Re_2}$$

The flux in the right leg is

$$\phi_2 = \frac{\mathcal{F}_2}{\Re_2} = \frac{\phi_1(\Re_{center}\|\Re_2)}{\Re_2} = \frac{\phi_1}{\Re_2}\frac{\Re_{center}\Re_2}{\Re_{center}+\Re_2} = \frac{\phi_1(\Re_3+\Re_4)}{\Re_3+\Re_4+\Re_2}$$

where

$$\Re_1 = \frac{l_a}{\mu_r\mu_0 A}, \Re_2 = \frac{l_b}{\mu_r\mu_0 A}, \Re_3 = \frac{2l_c}{\mu_r\mu_0 A}, \Re_4 = \frac{l_g}{\mu_0 A}$$

1.2 Eddy Current Loss, and Hyteresis Loss (Lecture 2)

Induced Voltage

Consider the magnetic circuit as shown in Figure 1.1, with the cross-sectional area A and the mean length l. Assume that the flux is a sinusoidal function of time, that is,

$$\phi(t) = \phi_{\max}\sin(\omega t) = AB_{\max}\sin(\omega t)$$

where $\phi_{\max}$ and $B_{\max}$ are the amplitudes of the flux and the flux density, respectively, and $\omega = 2\pi f$.

It follows from Faraday's law that the induced voltage is given by

$$e(t) = N\frac{d\phi}{dt} = N\omega\phi_{\max}\cos(\omega t) = E_{\max}\cos(\omega t)$$

where $E_{\max} = N\omega\phi_{\max} = \omega NAB_{\max} = 2\pi fNAB_{\max}$.

In steady-state operation, we are interested in rms values of voltages and currents. The rms value of the induced voltage is given by

$$E = \frac{2\pi}{\sqrt{2}}fNAB_{\max} = \sqrt{2}\,\pi fNAB_{\max}$$

Excitation Current

To produce a magnetic flux in a magnetic core, a current is required, which is referred to as the excitation current, denoted $i_\phi(t)$. Due to the nonlinearity of the B-H curve and the hysteresis property of the magnetic materials, $i_\phi(t) = \frac{Hl}{N}$ is not a sinusoidal function.

Eddy Current Loss

A time-varying flux induces an emf in the magnetic core in accordance with Faraday's law. Since the magnetic materials are good conductors, the induced emf produces a current along a closed path inside the magnetic core. Such a current is called eddy current because its swirling pattern resembles the eddy current of water.

As a consequence of this eddy current, energy is converted into heat in the resistance of the path, which gives rise to the power loss. Such a loss is referred to as the eddy current loss, which is determined by

$$P_e = k_e f^2 \delta^2 B_{\max}^2 V$$

where P_e is the eddy-current loss in watts (W), k_e is a constant that depends on the conductivity of the magnetic material, f is the frequency in hertz (Hz), δ is the lamination thickness in meters (m), $B_{\max}$ is the maximum flux density in teslas (T), and V is the volume of the magnetic material in cubic meters (m^3).

To reduce the effects of eddy currents, magnetic structures are usually built of thin sheets of laminations of the magnetic material, insulated from each other by an oxide layer or by a thin coat of insulation materials.

Hysteresis Loss

Assume that the flux in the core is initially zero. An AC current is applied to the winding. As the current increases for the first time, the flux in the core traces out path ab as shown in Figure 1.4. However, when the current decreases, the flux traces out a different path bcd, and later when the current increases again, the flux traces out path deb. This failure to retrace flux paths is called hysteresis. The path $bcdeb$ is called a hysteresis loop.

Each time the magnetic material is made to traverse its hysteresis loop, it produces a power loss, which is commonly referred to as the hysteresis loss. The hysteresis loss can be determined by

$$P_h = k_h f B_{\max}^n V$$

where P_h is the hysteresis loss in watts (W), k_h is a constant that depends on the magnetic material, and n is the Steinmetz exponent.

Core Loss

It is a common practice to lump the eddy current loss and hysteresis loss together to define the core loss

$$P_{core} = P_e + P_h = k_e f^2 \delta^2 B_{\max}^2 V + k_h f B_{\max}^n V = K_e f^2 B_{\max}^2 + K_h f B_{\max}^n$$

where $K_e = k_e \delta^2 V$ and $K_h = k_h V$.

1.3 Flux Linkage, Inductance, and Mutual Inductance

Inductance

Consider the magnetic circuit as shown in Figure 1.1. Faraday's law states that if a flux ϕ passes through a winding of N turns, a voltage will be induced in the winding and the induced voltage e is directly proportional to the rate of change in the flux linkages $\lambda = N\phi$ with respect to time, that is,

$$e = -\frac{d\lambda}{dt} = -N\frac{d\phi}{dt}$$

where the minus sign means that the polarity of the induced voltage is such that if the winding ends were short-circuited, it would produce current that would cause a flux opposing the original flux change.

The self inductance or inductance (in H) of the winding is defined as the ratio of the flux linkages and the current, that is,

$$L = \frac{\lambda}{i} = N\frac{\phi}{i}$$

If L is constant, then

$$e = -\frac{d\lambda}{dt} = -\frac{d(Li)}{dt} = -L\frac{di}{dt}$$

L depends on the physical dimensions of the magnetic circuit and the permeability of the magnetic materials. For the magnetic circuit as shown in Figure 1.1, L can be determined as follows:

$$L = \frac{\lambda}{i} = N\frac{\phi}{i} = N\frac{\frac{\mathcal{F}}{\Re}}{i} = N\frac{N}{\Re} = \frac{N^2}{\Re}$$

Example 1.3: The magnetic circuit of Figure 1.4 consists of an N-turn winding on a magnetic core of infinite permeability with two parallel air gaps of lengths g_1 and g_2 and areas A_1 and A_2, respectively. Find the inductance of the winding and the flux density B_1 in gap 1 when the winding is carrying a current i. Neglect fringing effects at the air gaps.

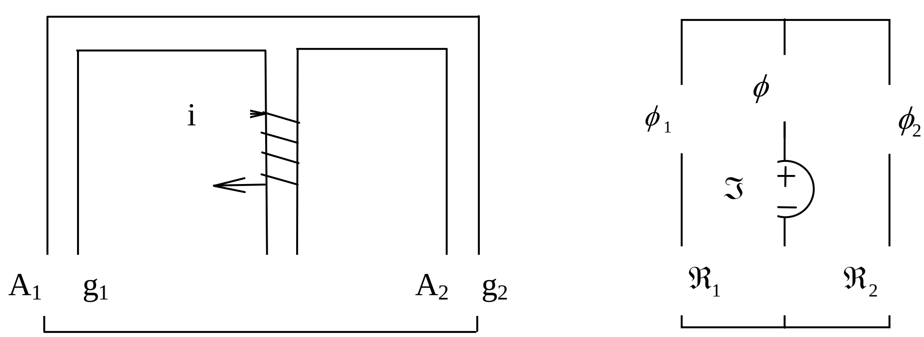

Figure 1.4

Solution: The equivalent circuit shows that the total reluctance is equal to the parallel combination of the two gap reluctances $\Re_1 = \frac{g_1}{\mu_0 A_1}$ and $\Re_2 = \frac{g_2}{\mu_0 A_2}$. Thus

$$\lambda = N\phi = N\frac{\mathcal{F}}{\Re} = N\frac{Ni}{\frac{\Re_1\Re_2}{\Re_1+\Re_2}} = \frac{N^2 i(\Re_1+\Re_2)}{\Re_1\Re_2}$$

and

$$L = \frac{\lambda}{i} = \frac{N^2(\Re_1+\Re_2)}{\Re_1\Re_2} = \frac{N^2\left(\frac{g_1}{\mu_0 A_1}+\frac{g_2}{\mu_0 A_2}\right)}{\frac{g_1}{\mu_0 A_1}\frac{g_2}{\mu_0 A_2}} = \mu_0 N^2\left(\frac{A_1}{g_1}+\frac{A_2}{g_2}\right)$$

The flux in gap 1 is

$$\phi_1 = \frac{Ni}{\Re_1} = \frac{\mu_0 A_1 Ni}{g_1}$$

and thus

$$B_1 = \frac{\phi_1}{A_1} = \frac{\mu_0 Ni}{g_1}$$

Mutual Inductance (Lecture 3)

Consider the magnetic circuit as shown in Figure 1.5. If a current i_1 is applied to coil-1 while a current i_2 to coil-2, then the total mmf is

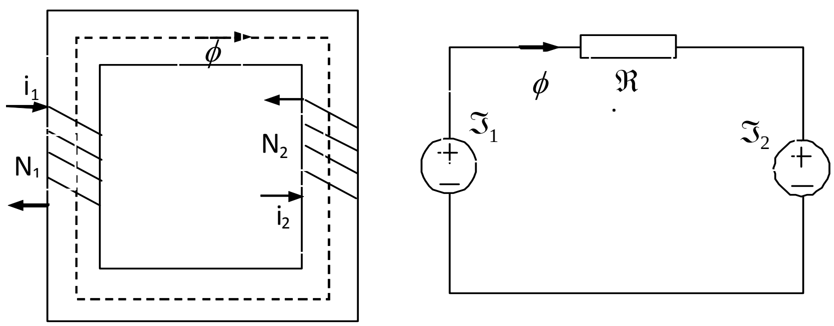

Figure 1.5

$$\mathcal{F} = N_1 i_1 + N_2 i_2$$

The reluctance of the core is

$$\Re = \frac{l}{\mu A}$$

The flux in the core is given by

$$\phi = \frac{\mathcal{F}}{\Re} = \frac{N_1}{\Re} i_1 + \frac{N_2}{\Re} i_2 = N_1 \frac{\mu A}{l} i_1 + N_2 \frac{\mu A}{l} i_2$$

The flux linkage of coil-1 is

$$\lambda_1 = N_1\phi = N_1^2 \frac{\mu A}{l} i_1 + N_1 N_2 \frac{\mu A}{l} i_2$$

which can be written

$$\lambda_1 = L_{11} i_1 + L_{12} i_2 = \lambda_{11} + \lambda_{12}$$

where

$$L_{11} = \frac{\lambda_{11}}{i_1} = N_1^2 \frac{\mu A}{l}$$

is the self-inductance of coil 1 and $\lambda_{11} = L_{11} i_1$ is the flux linkage of coil-1 due to its own current i_1. The mutual inductance from coil-2 to coil-1 is

$$L_{12} = \frac{\lambda_{12}}{i_2} = N_1 N_2 \frac{\mu A}{l}$$

and $\lambda_{12} = L_{12} i_2$ is the flux linkage of coil-1 due to the current i_2.
Similarly, the flux linkage of coil-2 is

$$\lambda_2 = N_2\phi = N_1 N_2 \frac{\mu A}{l} i_1 + N_2^2 \frac{\mu A}{l} i_2 = L_{21} i_1 + L_{22} i_2 = \lambda_{21} + \lambda_{22}$$

with $L_{21} = \frac{\lambda_{21}}{i_1} = L_{12} = N_1 N_2 \frac{\mu A}{l}$, $L_{22} = \frac{\lambda_{22}}{i_2} = N_2^2 \frac{\mu A}{l}$, $\lambda_{21} = L_{21} i_2$, and $\lambda_{22} = L_{22} i_2$.
Now suppose $\lambda_{12} = k_1 \lambda_{11}$ and $\lambda_{21} = k_2 \lambda_{22}$. Then it is easily checked that

$$L_{12} L_{21} = \frac{\lambda_{12}}{i_2}\frac{\lambda_{21}}{i_1} = \frac{k_1 \lambda_{11}}{i_2}\frac{k_2 \lambda_{22}}{i_1} = k_1 k_2 L_{11} L_{22}$$

In a linear system, $L_{12} = L_{21} = M$. Therefore,

$$M = k\sqrt{L_{11}L_{22}}$$

where $k = \sqrt{k_1 k_2}$ is known as the coefficient of coupling or the coupling factor between the two coils.

If the inductances are constant, then the induced voltages can be calculated by

$$e_1 = \frac{d\lambda_1}{dt} = \frac{d(L_{11}i_1+L_{12}i_2)}{dt} = L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt}$$

$$e_2 = \frac{d\lambda_2}{dt} = \frac{d(L_{21}i_1+L_{22}i_2)}{dt} = L_{21}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt}$$

Example 1.4: A magnetic circuit with two windings was tested with an AC source at $60Hz$ and the following data were recorded.

Test	Coil Condition	RMS Voltage (V)	RMS Current (A)
1	Coil-1 connected to a voltage source	80	1.5
	Coil-2 open circuit	30	0
2	Coil-2 connected to a voltage source	60	1.0
	Coil-1 open circuit	20	0

Assume the magnetic circuit operated in the linear region and neglect the hysteresis effects. Neglect the winding resistances. Determine the self inductance, mutual inductance, and coupling factor.

Solution: The AC currents can be expressed by $i_j(t) = \sqrt{2}\,I_j\cos(\omega t)$ with $\omega = 2\pi f$ and $j = 1, 2$. For the first test, the following equations are obtained:

$$v_1 = e_1 = L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} = L_{11}\frac{di_1}{dt} = -\sqrt{2}\,\omega L_{11}I_1\sin(\omega t)$$

$$v_2 = e_2 = L_{21}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt} = L_{21}\frac{di_1}{dt} = -\sqrt{2}\,\omega L_{21}I_1\sin(\omega t)$$

which implies that the rms values of v_1 and v_2 are equal to $\omega L_{11}I_1$ and $\omega L_{21}I_1$, that is,

$$V_1 = \omega L_{11}I_1 \Rightarrow L_{11} = \frac{V_1}{\omega I_1} = \frac{80}{2\pi\times 60\times 1.5} = 0.14147H$$

$$V_2 = \omega L_{21}I_1 \Rightarrow L_{21} = \frac{V_2}{\omega I_1} = \frac{30}{2\pi\times 60\times 1.5} = 53.05\,mH$$

Similarly, it follows from the second test that

$$v_1 = e_1 = L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} = L_{12}\frac{di_2}{dt} = -\sqrt{2}\,\omega L_{21}I_2\sin(\omega t)$$

$$v_2 = e_2 = L_{21}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt} = L_{22}\frac{di_2}{dt} = -\sqrt{2}\,\omega L_{22}I_2\sin(\omega t)$$

and

$$V_1 = \omega L_{12}I_2 \Rightarrow L_{12} = \frac{V_1}{\omega I_2} = \frac{20}{2\pi\times 60\times 1.0} = 53.05mH$$

$$V_2 = \omega L_{22}I_2 \Rightarrow L_{22} = \frac{V_2}{\omega I_2} = \frac{60}{2\pi\times 60\times 1.0} = 0.15915H$$

The coupling factor is

$$k = \frac{L_{12}}{\sqrt{L_{11}L_{22}}} = \frac{53.05\times 10^{-3}}{\sqrt{0.14147\times 0.15915}} = 0.35355$$

Example 1.5: Given the magnetic circuit as shown in Figure 1.6, neglect fringing effects, leakage flux and reluctances in the magnetic materials. Determine the self-inductances and mutual inductances.

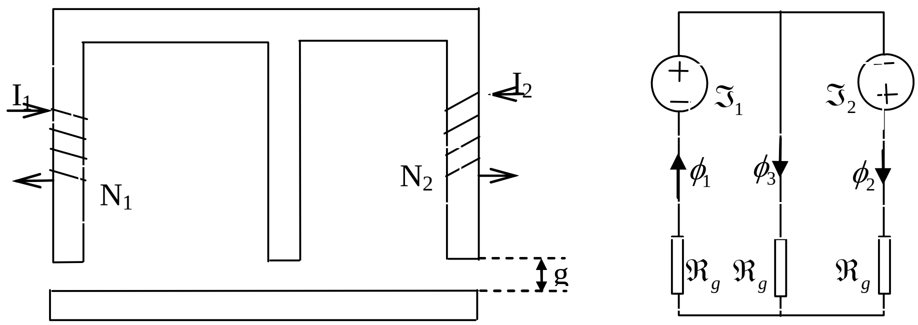

Figure1.6 The magnetic circuit for Example 1.5

Solution: The reluctance of each air gap is $\Re_g = \frac{g}{\mu_0 A}$ where A is the cross-sectional area of the gap. The fluxes satisfy the following equation

$$\phi_1 = \phi_2 + \phi_3$$

For the left loop, we have

$$\mathcal{F}_1 = \phi_1 \Re_g + \phi_3 \Re_g = \Re_g(\phi_1 + \phi_3)$$

that is,

$$\phi_1 + \phi_3 = \frac{\mathcal{F}_1}{\Re_g}$$

For the right loop, we have

$$\mathcal{F}_2 = \phi_2 \Re_g - \phi_3 \Re_g = \Re_g(\phi_2 - \phi_3)$$

that is,

$$\phi_2 - \phi_3 = \frac{\mathcal{F}_2}{\Re_g}$$

Substituting $\phi_1 = \phi_2 + \phi_3$ into $\phi_1 + \phi_3 = \frac{\mathcal{F}_1}{\Re_g}$ gives

$$\phi_2 + 2\phi_3 = \frac{\mathcal{F}_1}{\Re_g}$$

Subtracting $\phi_2 - \phi_3 = \frac{\mathcal{F}_2}{\Re_g}$ from $\phi_2 + 2\phi_3 = \frac{\mathcal{F}_1}{\Re_g}$ yields

$$3\phi_3 = \frac{\mathcal{F}_1}{\Re_g} - \frac{\mathcal{F}_2}{\Re_g} = \frac{\mathcal{F}_1 - \mathcal{F}_2}{\Re_g}$$

that is,

$$\phi_3 = \frac{\mathcal{F}_1 - \mathcal{F}_2}{3\Re_g}$$

Then, it follows from $\phi_2 - \phi_3 = \frac{\mathcal{F}_2}{\Re_g}$ that

$$\phi_2 = \phi_3 + \frac{\mathcal{F}_2}{\Re_g} = \frac{\mathcal{F}_1 - \mathcal{F}_2}{3\Re_g} + \frac{3\mathcal{F}_2}{3\Re_g} = \frac{\mathcal{F}_1 + 2\mathcal{F}_2}{3\Re_g} = \frac{N_1 i_1 + 2N_2 i_2}{3\Re_g}$$

and from $\phi_1 = \phi_2 + \phi_3$, we have

$$\phi_1 = \phi_2 + \phi_3 = \frac{\mathcal{F}_1 + 2\mathcal{F}_2}{3\Re_g} + \frac{\mathcal{F}_1 - \mathcal{F}_2}{3\Re_g} = \frac{2\mathcal{F}_1 + \mathcal{F}_2}{3\Re_g} = \frac{2N_1 i_1 + N_2 i_2}{3\Re_g}$$

Therefore,

$$\lambda_1 = N_1 \phi_1 = \frac{2N_1^2}{3\Re_g} i_1 + \frac{N_1 N_2}{3\Re_g} i_2$$

$$\lambda_2 = N_2 \phi_2 = \frac{N_1 N_2}{3\Re_g} i_1 + \frac{2N_2^2}{3\Re_g} i_2$$

which implies that

$$L_{11} = \frac{2N_1^2}{3\Re_g}, L_{12} = L_{21} = \frac{N_1 N_2}{3\Re_g}, L_{22} = \frac{2N_2^2}{3\Re_g}$$

with $\Re_g = \frac{g}{\mu_0 A}$.

Chapter 2. Electromechanical Energy Conversion

An electromechanical system consists of an electrical subsystem (electric circuits such as windings), a magnetic subsystem (magnetic field in the magnetic cores and air-gaps), and a mechanical subsystem (mechanically movable parts such as a plunger in a linear actuator and a rotor in a rotating electrical machine), as shown in Figure 2.1.

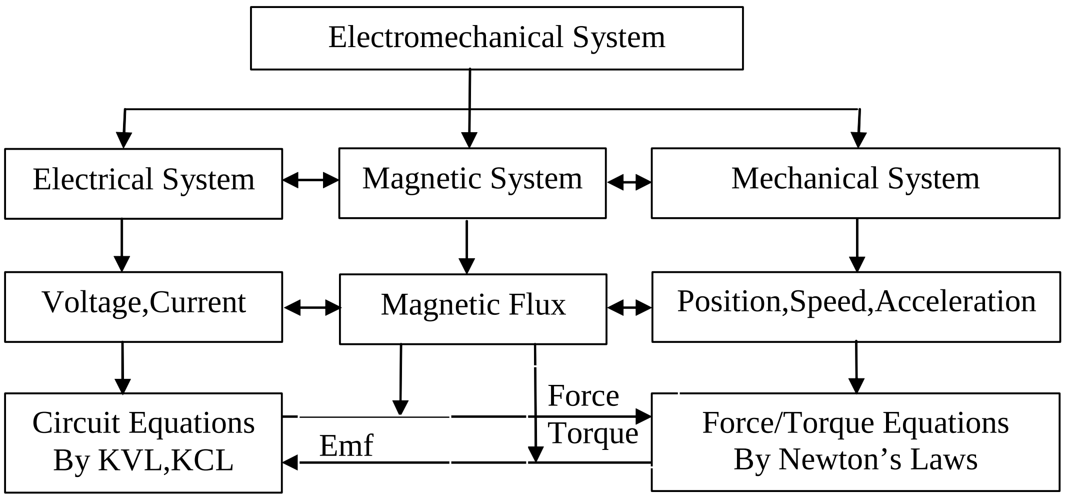

Figure 2.1 General concept of electromechanical system modeling

2.1 Force and Torque on a Current Carrying Conductor: Motor Action (Lecture 4)

The force on a conductor carrying a current i in a uniform magnetic field $\vec{B}$ is given by the Lorentz's force law:

$$\vec{f} = i\vec{l} \times \vec{B} = ilB\sin\theta$$

In a rotating system, the torque about an axis can be calculated by

$$\tau = \vec{r} \times \vec{f}$$

where $\vec{r}$ is the radius vector from the axis towards the conductor.

Right Hand Rule for Cross Product: When the thumb of the right hand points in the direction of the first vector and the index finger points in the direction of the second vector, the cross product, which is perpendicular to the directions of both vectors, points in the direction normal to the palm of the hand.

2.2 Energy Stored in Magnetic Field

Energy Stored in Magnetic Circuit with a Single Coil

Consider the magnetic circuit with a single winding as shown in Figure 1.1. Neglect losses. Note that

$$e = \frac{d\lambda}{dt}$$

and

$$L = \frac{\lambda}{i}$$

The electric input power is determined from

$$p = ie = i\frac{d\lambda}{dt}$$

The energy stored in the field during dt is

$$dW_\phi = pdt = id\lambda$$

With zero initial energy stored in the magnetic field, the energy at time t is

$$W_\phi = \int_0^t pdt = \int_0^\lambda id\lambda = \int_0^\lambda \frac{\lambda}{L}d\lambda = \frac{1}{2L}\lambda^2$$

or

$$W_\phi = \int_0^t pdt = \int_0^\lambda id\lambda = \int_0^i id(Li) = \frac{1}{2}Li^2$$

Example 2.1: (see Example 1.1) A magnetic core is shown in Figure 1.3. Both width and depth are 3cm and its mean length is 30cm. The length of the air-gap is 0.05cm. The coil has 500 turns. The relative permeability of the core is assumed to be 70,000. Neglect fringing effects and assume the flux density of the core is $B_c = 1.0Wb/m^2$. The frequency of the source is $60Hz$. Find the inductances of the core and energy stored in the field.

Solution: It follows from Example 1.1 that

$$\Re_c = \frac{l_c}{\mu_r\mu_0 A_c} = \frac{30\times10^{-2}}{70000\times4\pi\times10^{-7}\times0.03\times0.03} = 3789.4A\cdot t/Wb$$

$$\Re_a = \frac{l_a}{\mu_0 A_a} = \frac{0.05\times10^{-2}}{4\pi\times10^{-7}\times0.03\times0.03} = 442100A\cdot t/Wb$$

$$\Re = \Re_c + \Re_a = 3789.4 + 442100 = 445890A\cdot t/Wb$$

The inductance is

$$L = \frac{\lambda}{i} = \frac{N\phi}{i} = \frac{N}{i}\frac{\mathcal{F}}{\Re} = \frac{N}{i}\frac{Ni}{\Re} = \frac{N^2}{\Re} = \frac{500^2}{445890} = 0.56068H$$

If $\Re_c$ is neglected,

$$L = \frac{N^2}{\Re_a} = \frac{500^2}{442100} = 0.56548H$$

The error caused by neglecting the reluctance of the core is only

$$error = 0.56548 - 0.56068 = 0.0048H.$$

The flux in the magnetic circuit is

$$\phi = B_c A_c = 1.0\times0.03\times0.03 = 0.0009Wb$$

The current in the coil is

$$i = \frac{\phi\Re}{N} = \frac{0.0009\times445890}{500} = 0.8026A$$

The stored energy is

$$W_\phi = \frac{1}{2}Li^2 = \frac{1}{2}\times0.56068\times(0.8026)^2 = 0.18059J$$

Energy Stored in Magnetic Circuit with Two Coils

Consider the magnetic circuit with two windings as shown in Figure 1.5. Neglect losses. Then the electric input energy is equal to the energy stored in the field, that is,

$$dW_s = dW_\phi$$

The electric input power is

$$p = e_1 i_1 + e_2 i_2$$

and the input energy is

$$dW_e = pdt = e_1 i_1 dt + e_2 i_2 dt$$

Note that $e_1 = \frac{d\lambda_1}{dt}$ and $e_2 = \frac{d\lambda_2}{dt}$.

Thus, the stored energy can be expressed as

$$dW_\phi = dW_e = i_1 d\lambda_1 + i_2 d\lambda_2$$

Recall the relations $L_{12} = L_{21}$ and

$$\lambda_1 = \lambda_{11} + \lambda_{12} = L_{11} i_1 + L_{12} i_2$$

$$\lambda_2 = \lambda_{21} + \lambda_{22} = L_{21} i_1 + L_{22} i_2$$

to get

$$\begin{aligned} dW_\phi = dW_e &= i_1 d(L_{11} i_1 + L_{12} i_2) + i_2 d(L_{21} i_1 + L_{22} i_2) \\ &= L_{11} i_1 di_1 + L_{12} i_1 di_2 + L_{21} i_2 di_1 + L_{22} i_2 di_2 \\ &= L_{11} i_1 di_1 + L_{12}(i_1 di_2 + i_2 di_1) + L_{22} i_2 di_2 \\ &= L_{11} i_1 di_1 + L_{12} d(i_1 i_2) + L_{22} i_2 di_2 \end{aligned}$$

For the case that the inductances are independent of currents, W_ϕ can be calculated by

$$W_\phi = \int (L_{11} i_1 di_1 + L_{12} d(i_1 i_2) + L_{22} i_2 di_2) = \frac{1}{2} L_{11} i_1^2 + L_{12} i_1 i_2 + \frac{1}{2} L_{22} i_2^2$$

On the other hand, solving the equations

$$\lambda_1 = L_{11} i_1 + L_{12} i_2$$

$$\lambda_2 = L_{21} i_1 + L_{22} i_2$$

for i_1 and i_2 yields

$$i_1 = \Gamma_{11} \lambda_1 + \Gamma_{12} \lambda_2$$

$$i_2 = \Gamma_{21} \lambda_1 + \Gamma_{22} \lambda_2$$

where $\Gamma_{11} = L_{22}/\Delta, \Gamma_{12} = \Gamma_{21} = -L_{12}/\Delta, \Gamma_{22} = L_{11}/\Delta,$ and $\Delta = L_{11} L_{22} - (L_{12})^2$.
Then,

$$\begin{aligned} dW_\phi &= (\Gamma_{11} \lambda_1 + \Gamma_{12} \lambda_2) d\lambda_1 + (\Gamma_{21} \lambda_1 + \Gamma_{22} \lambda_2) d\lambda_2 \\ &= \Gamma_{11} \lambda_1 d\lambda_1 + \Gamma_{12} \lambda_2 d\lambda_1 + \Gamma_{21} \lambda_1 d\lambda_2 + \Gamma_{22} \lambda_2 d\lambda_2 \\ &= \Gamma_{11} \lambda_1 d\lambda_1 + \Gamma_{12}(\lambda_2 d\lambda_1 + \lambda_1 d\lambda_2) + \Gamma_{22} \lambda_2 d\lambda_2 \\ &= \Gamma_{11} \lambda_1 d\lambda_1 + \Gamma_{12} d(\lambda_1 \lambda_2) + \Gamma_{22} \lambda_2 d\lambda_2 \end{aligned}$$

which means that

$$W_\phi = \int (\Gamma_{11} \lambda_1 d\lambda_1 + \Gamma_{12} d(\lambda_2 \lambda_1) + \Gamma_{22} \lambda_2 d\lambda_2) = \frac{1}{2} \Gamma_{11}^2 \lambda_1^2 + \Gamma_{12} \lambda_1 \lambda_2 + \frac{1}{2} \Gamma_{22} \lambda_2^2$$

2.3 Force and Torque Calculation from Energy (Lecture 5)

A Singly Excited Linear Actuator

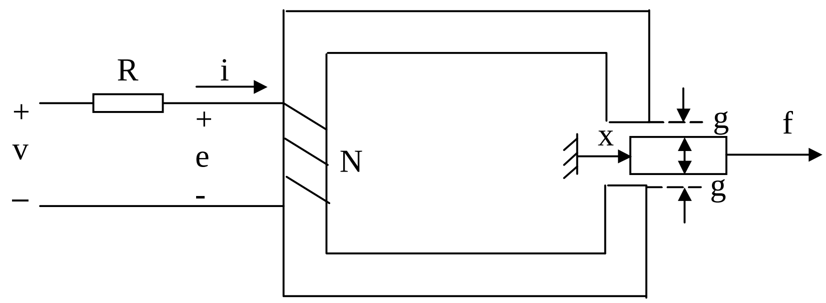

Figure 2.2

Consider a singly excited linear actuator as shown in Figure 2.2. The winding resistance is R. A voltage v is applied to the winding, which produces a current i. Assume that at a certain time instant t, the movable plunger is positioned at x and the force acting on the plunger is $\vec{f}$ with the reference direction chosen in the positive direction of the x axis, as shown in the diagram. After a time interval dt, the plunger has moved for a distance dx under the action of the force $\vec{f}$. The mechanical work done by the force during this time interval is thus

$$dW_m = fdx$$

The electrical energy supplied by the electrical source during this time interval is calculated by

$$dW_e = vidt$$

The energy dissipated in the winding resistance during this time interval is

$$dW_{loss} = Ri^2dt$$

Suppose that there is no mechanical losses in the system. According to the principle of conservation of energy (energy is neither created nor destroyed and it is merely changed in form), the energy stored in the magnetic field during this time interval dW_ϕ must satisfy

$$dW_\phi = dW_e - dW_{loss} - dW_m = vidt - Ri^2dt - fdx = (vi - Ri)idt - fdx = eidt - fdx = \frac{d\lambda}{dt}idt - fdx$$

From the above equation, we know that the energy stored in the magnetic field W_ϕ is a function of λ and x. Therefore, W_ϕ can be expressed as

$$W_\phi(\lambda, x) = \frac{\partial W_\phi(\lambda,x)}{\partial \lambda}d\lambda + \frac{\partial W_\phi(\lambda,x)}{\partial x}dx$$

By comparing the above two equations, we get

$$i = \frac{\partial W_\phi(\lambda,x)}{\partial \lambda}, f = -\frac{\partial W_\phi(\lambda,x)}{\partial x}$$

It follows from Section 2.2 that the energy stored in the magnetic field can be calculated by

$$W_\phi(\lambda, x) = \int_0^\lambda i(\lambda, x)d\lambda$$

For a magnetically linear system (with a constant permeability or a straight line magnetization curve such that the inductance of the coil is independent of the excitation current), the above expression becomes

$$W_\phi(\lambda, x) = \frac{1}{2}\frac{\lambda^2}{L(x)}$$

Therefore, the force can be calculated by

$$f = -\frac{\partial W_\phi(\lambda, x)}{\partial x} = \frac{1}{2}\left(\frac{\lambda}{L(x)}\right)^2 \frac{dL(x)}{dx} = \frac{1}{2} i^2 \frac{dL(x)}{dx}$$

Example 2.2: Calculate the force acting on the plunger of a linear actuator as shown in Figure 2.2, where the magnetic core has infinite relative permeability and fringing effects are negligible.

Solution: The reluctance of the actuator is

$$\Re_g = \frac{2g}{\mu_0 (d-x) l}$$

The inductance of the actuator is

$$L(x) = \frac{N^2}{\Re_g} = \frac{\mu_0 N^2 l}{2g}(d-x)$$

Therefore, the force acting on the plunger is

$$f = \frac{1}{2} i^2 \frac{dL(x)}{dx} = -\frac{\mu_0 l}{4g}(Ni)^2$$

The minus sign of the force indicates that the direction of the force is to reduce the displacement so as to reduce the reluctance of the air gaps. Since this force is caused by the variation of magnetic reluctance of the magnetic circuit, it is known as the reluctance force.

Doubly Excited Rotating Actuator

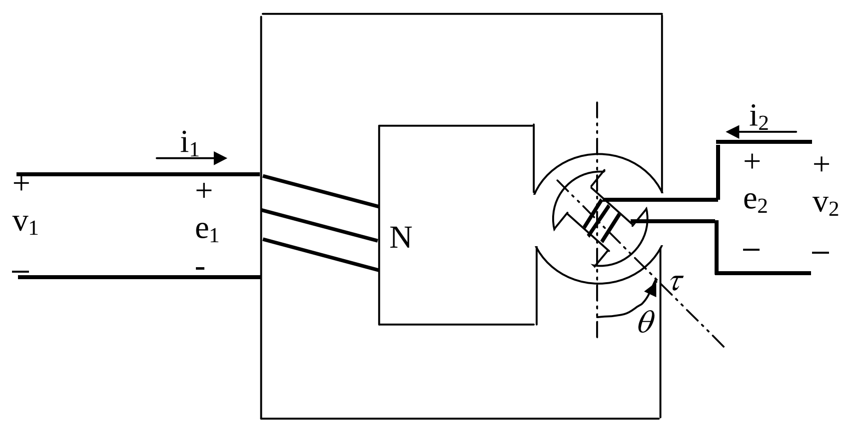

Figure 2.3

The general principle for force and torque calculation discussed above is equally applicable to multi-excited systems. Consider a doubly excited rotating actuator as shown in Figure 2.3. The differential energy functions can be derived as following:

$$dW_\phi = dW_e - dW_m$$

where

$$dW_e = i_1 d\lambda_1 + i_2 d\lambda_2$$
$$dW_m = \tau d\theta$$

Hence,

$$dW_\phi(\lambda_1, \lambda_2, \theta) = i_1 d\lambda_1 + i_2 d\lambda_2 - \tau d\theta = \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \lambda_1} d\lambda_1 + \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \lambda_2} d\lambda_2 + \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \theta} d\theta$$

which implies that

$$i_1 = \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \lambda_1}$$

$$i_2 = \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \lambda_2}$$
$$\tau = -\frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \theta}$$
Note that for a linear system
$$W_\phi(\lambda_1,\lambda_2,\theta) = \frac{1}{2}\Gamma_{11}^2\lambda_1^2 + \Gamma_{12}\lambda_1\lambda_2 + \frac{1}{2}\Gamma_{22}\lambda_2^2$$
Then, we have
$$\tau = -\frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \theta} = -\left(\frac{1}{2}\lambda_1^2\frac{d\Gamma_{11}(\theta)}{d\theta} + \lambda_1\lambda_2\frac{d\Gamma_{12}(\theta)}{d\theta} + \frac{1}{2}\lambda_2^2\frac{d\Gamma_{22}(\theta)}{d\theta}\right)$$
It is useful to express τ in terms of L_{11}, L_{12}, and L_{22}. It can be verified that
$$\tau = \frac{\partial W_\phi(i_1,i_2,\theta)}{\partial \theta} = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1 i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$$
If there is only one coil in the magnetic circuit, the torque becomes
$$\tau = \frac{\partial W_\phi(i,\theta)}{\partial \theta} = \frac{1}{2}i^2\frac{dL(\theta)}{d\theta}$$
Example 2.3: Write an expression for the inductance of the magnetic circuit for Figure 2.4 as a function of θ and derive an expression for the torque acting on the robot as a function of the winding current i and the rotor angle θ. Neglect the effects of fringing and the reluctance of the steel. The redius of the rotor is r and the length of air gap is g.

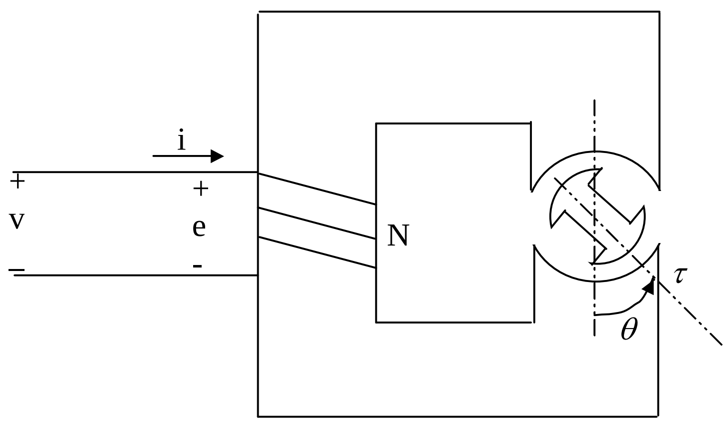

Figure 2.4

Solution: The reluctance of the air-gaps is
$$\Re_g = \frac{2g}{\mu_0 h(r+0.5g)\theta}$$
The inductance of the magnetic circuit is
$$L(\theta) = \frac{N^2}{\Re_g} = \frac{\mu_0 N^2 h(r+0.5g)}{2g}\theta$$
The energy stored in the magnetic field is
$$W_\phi(\lambda,\theta) = \int_0^\lambda i(\lambda,\theta)d\lambda = \int_0^\lambda \frac{\lambda}{L(\theta)}d\lambda = \frac{1}{2}\frac{\lambda^2}{L(\theta)}$$
The torque is
$$\tau = -\frac{\partial W_\phi(\lambda,\theta)}{\partial \theta} = \frac{1}{2}\left(\frac{\lambda}{L(\theta)}\right)^2\frac{dL(\theta)}{d\theta} = \frac{1}{2}i^2\frac{dL(\theta)}{d\theta} = \frac{1}{2}\frac{\mu_0 N^2 h(r+0.5g)}{2g}i^2$$
Example 2.4: In the system shown in Figure 2.3, the inductances in henries are given as
$$L_{11} = 0.001(3+\cos 2\theta)$$

$$L_{12} = 0.3\cos\theta$$
$$L_{22} = 30 + 10\cos 2\theta$$

Find the torque $\tau(\theta)$for currents $i_1 = 0.8A$ and $i_2 = 0.01A$.

Solution: The torque can be determined by

$$\tau = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1 i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$$
$$= \frac{1}{2}i_1^2(-0.002\sin 2\theta) + i_1 i_2(-0.3\sin\theta) + \frac{1}{2}i_2^2(-20\sin 2\theta)$$
$$= -0.001 i_1^2\sin 2\theta - 0.3 i_1 i_2\sin\theta - 10 i_2^2\sin 2\theta$$
$$= -0.001\times 0.8^2\sin 2\theta + 0.3\times 0.8\times 0.01\sin\theta - 10\times 0.01^2\sin 2\theta$$
$$= -0.00164\sin 2\theta - 0.0024\sin\theta$$

Example 2.5: The magnetic circuit of Figure 2.6 is excited by a 100-turn coil wound over the central leg. The depth is 1cm, a=1cm and b=5cm. Determine the current in the coil that is necessary to keep the movable part suspended at a distance of 1cm. Both magnetic circuit and movable part have a cross-sectional area of 1cm^2. What is the energy stored in the systems? The relative permeability and the density of the magnetic material are 2000 and 7.85g/cm^3, respectively.

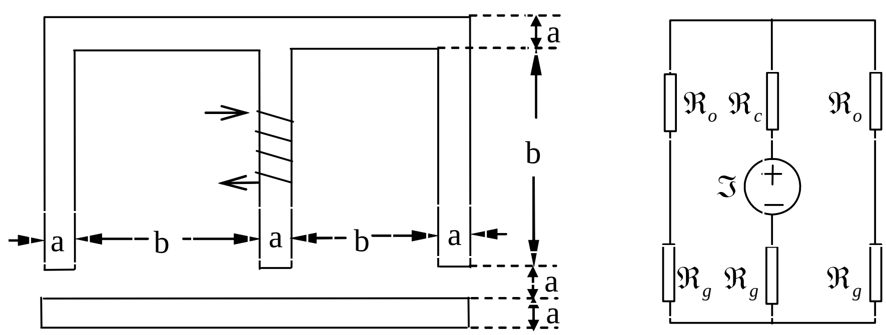

Figure2.6 Figure for Example 2.5

Solution: The mean length for each of the outer legs including a part of the movable part is

$$l_o = \frac{1}{2}a + b + \frac{1}{2}a + \frac{1}{2}a + b + \frac{1}{2}a + \frac{1}{2}a + b + \frac{1}{2}a = 3a + 3b = 3(a+b) = 3(1+5) = 18cm$$

The mean length of the central leg is

$$l_c = \frac{1}{2}a + b + \frac{1}{2}a = a + b = 1 + 5 = 6cm$$

The length of the air gap is assumed to be x. The reluctance of each par is calculated as

$$\Re_o = \frac{l_o}{\mu_r\mu_0 A} = \frac{18\times10^{-2}}{2000\times4\pi\times10^{-7}\times0.0001} = 7.1620\times10^5 A\cdot t/Wb$$
$$\Re_c = \frac{l_c}{\mu_r\mu_0 A} = \frac{6\times10^{-2}}{2000\times4\pi\times10^{-7}\times0.0001} = 2.3873\times10^5 A\cdot t/Wb$$
$$\Re_g = \frac{l_c}{\mu_r\mu_0 A} = \frac{x}{4\pi\times10^{-7}\times0.0001} = 7.9577\times10^9 x$$

The applied mmf is $\mathcal{F} = Ni = 100i$ where i is the required current in the coil.

The total reluctance as viewed from the magnetomotive source is

$$\begin{aligned}\Re &= \Re_c + \Re_g + 0.5(\Re_o + \Re_g) \\ &= 2.3873 \times 10^5 + 7.9577 \times 10^9 x + 0.5(7.1620 \times 10^5 + 7.9577 \times 10^9 x) \\ &= 1.1937 \times 10^{10} x + 5.9683 \times 10^5\end{aligned}$$

Hence, the inductance is

$$L(x) = \frac{N^2}{\Re} = \frac{100^2}{1.1937\times10^{10}x+5.9683\times10^5} = \frac{1}{1.1937\times10^6 x+59.683}$$

The magnetic force acting on the movable part is

$$f = \frac{1}{2}i^2\frac{dL(x)}{dx} = -\frac{1}{2}i^2\frac{1.1937\times10^6}{(59.683+1.1937\times10^6 x)^2}$$

The negative sign indicates that the force is acting in the upward direction. Therefore, the magnitude of the force of attraction for $x = 1cm$ is

$$f = -\frac{1}{2}\frac{1.1937\times10^6}{(59.683+1.1937\times10^6\times0.01)^2}i^2 = 4.1471\times10^{-3}i^2$$

The length of the movable part is $3a + 2b = 13cm$. The volume of the movable part is $13 \times 1 = 13cm^3$, so the mass of the movable part is $13 \times 7.85 = 102.05g$.

For the movable part to be stationary, the force of gravity must equal to the magnetic force calculated by

$$f_g = mg = 102.05 \times 10^{-3} \times 9.8 = 1.0001N$$

that is

$$4.1471 \times 10^{-3} i^2 = 1.0001$$

Solving this equation for the current gives

$$i = \sqrt{\frac{1.0001}{4.1471\times10^{-3}}} = 15.529A$$

The inductance of the magnetic circuit at $x = 1cm$ is

$$L(1cm) = \frac{1}{1.1937\times10^6\times0.01+59.683} = 8.3356\times10^{-5}H$$

The energy stored in the magnetic field is

$$W_\phi = \frac{1}{2}Li^2 = \frac{1}{2}\times 8.3356\times10^{-5}\times15.529^2 = 1.0051\times10^{-2}J$$

Chapter 3 Dynamics of Electromechanical Systems

3.1 Mathematical Model

Figure 3.1 shows the model of a simple electromechanical system, which consists of three parts: an electrical system, an electromechanical energy-conversion system, and an mechanical system.

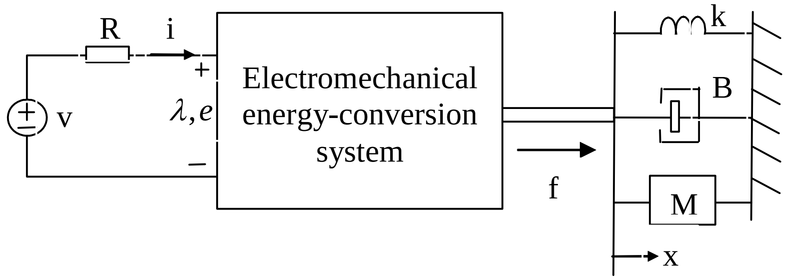

Figure3.1 Model of an electromechanical system

Neglect losses in the electromechanical system. For the electrical system, the following equation can be obtained from KVL:

$$v = Ri + e = Ri + \frac{d\lambda}{dt} = Ri + \frac{d(L(x)i)}{dt} = Ri + L(x)\frac{di}{dt} + i\frac{dL(x)}{dx}\frac{dx}{dt}$$

Assume that the spring is normally unstretched at $x = 0$. Then, the following equation can be obtained from Newton's law:

$$f - kx - B\frac{dx}{dt} = M\frac{d^2x}{dt^2}$$

where f and $L(x)$ depend on the properties of the electromechanical energy-conversion system.

The differential equations above are called the mathematical model of the electromechanical system.

Example 3.1: An electromechanical system is shown in Figure 3.2. The voltage source has a DC voltage V_s. The switch is turned on at $t = 0$. The bar slides along a pair of frictionless rails in a horizontal plane. The bar has a mass of m. The resistance of the system is R. Assume all initial conditions are zero. Determine the current $i(t)$ and the velocity $v = \frac{dx}{dt}$ of the bar.

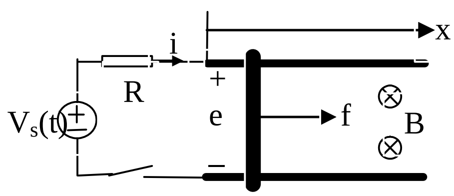

Figure3.2 Example 3.1

Solution: The induced voltage is

$$e(t) = l\vec{v} \times \vec{B} = lBv(t)$$

From KVL, we obtain

$$v_s(t) = Ri(t) + e(t) = Ri(t) + lBv(t)$$

which implies that

$$i(t) = \frac{1}{R}v_s(t) - \frac{lB}{R}v(t)$$

The induced force is

$$f = i\vec{l} \times \vec{B} = lBi(t) = \frac{lB}{R}v_s(t) - \frac{l^2B^2}{R}v(t)$$

From Newton's law, we have

$$f = \frac{lB}{R}v_s(t) - \frac{l^2B^2}{R}v(t) = m\frac{dv(t)}{dt}$$

So the mathematical model for this system is

$$m\frac{dv(t)}{dt} + \frac{l^2B^2}{R}v(t) = \frac{lB}{R}v_s(t)$$

This equation can be solved by using Laplace transform. Note that $v_s(t)$ is a step signal and its Laplace transform is $\frac{V_s}{s}$. Taking Laplace transform gives

$$msV(s) + \frac{l^2B^2}{R}V(s) = \frac{lB}{R}\frac{V_s}{s}$$

Solving it for $V(s)$ yields

$$V(s) = \frac{\frac{lB}{R}\frac{V_s}{s}}{ms+\frac{l^2B^2}{R}} = \frac{\frac{lBV_s}{mR}}{s\left(s+\frac{l^2B^2}{mR}\right)}$$

Carrying the partial fraction expansion gives

$$V(s) = \frac{A_1}{s} + \frac{A_2}{s+\frac{l^2B^2}{mR}}$$

where

$$A_1 = s\frac{\frac{lBV_s}{mR}}{s\left(s+\frac{l^2B^2}{mR}\right)}\Bigg|_{s=0} = \frac{\frac{lBV_s}{mR}}{\frac{l^2B^2}{mR}} = \frac{V_s}{lB}$$

$$A_2 = \left(s+\frac{l^2B^2}{mR}\right)\frac{\frac{lBV_s}{mR}}{s\left(s+\frac{l^2B^2}{mR}\right)}\Bigg|_{s=-\frac{l^2B^2}{mR}} = \frac{\frac{lBV_s}{mR}}{-\frac{l^2B^2}{mR}} = -\frac{V_s}{lB}$$

Therefore,

$$V(s) = \frac{\frac{V_s}{lB}}{s} + \frac{-\frac{V_s}{lB}}{s+\frac{l^2B^2}{mR}}$$

Taking inverse Laplace transform gives

$$v(t) = \frac{V_s}{lB} - \frac{V_s}{lB}e^{-\frac{l^2B^2}{mR}t}$$

The current in the circuit is given by

$$i(t) = \frac{1}{R}v_s(t) - \frac{lB}{R}v(t) = \frac{V_s}{R} - \frac{lB}{R}\left(\frac{V_s}{lB} - \frac{V_s}{lB}e^{-\frac{l^2B^2}{mR}t}\right) = \frac{V_s}{R}e^{-\frac{l^2B^2}{mR}t}$$

3.2 Dynamics of DC Generators

A separately excited DC generator delivering power to a static load is shown in Figure 3.3. Assume that the speed of the generator is constant.

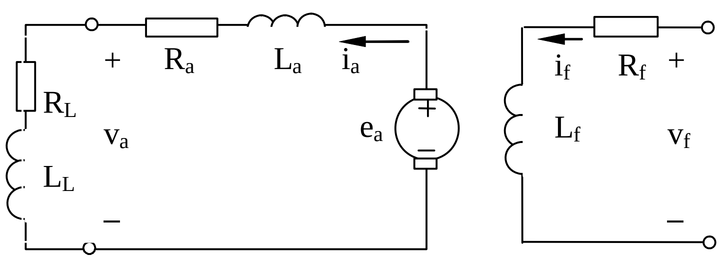

Figure3.3 Equivalent circuit of a dc generator

During the transient state, the field voltage satisfies the equation

$$V_f = R_f i_f(t) + L_f\frac{di_f(t)}{dt}$$

and the generated voltage is

$$e_a(t) = K_e\omega i_f(t) = (R_a + R_L)i_a(t) + (L_a + L_L)\frac{di_a(t)}{dt}$$

Taking the Laplace transform gives

$$V_f(s) = R_f I_f(s) + L_f[sI_f(s) - i_f(0)]$$
$$K_e \omega I_f(s) = (R_a + R_L)I_a(s) + (L_a + L_L)[sI_a(s) - i_a(0)]$$
Solving these equations yields
$$I_f(s) = \frac{V_f(s)+L_f i_f(0)}{L_f s+R_f}$$
$$I_a(s) = \frac{K_e \omega I_f(s)+(L_a+L_L)i_a(0)}{(L_a+L_L)s+R_a+R_L} = \frac{K_e \omega (V_f(s)+L_f i_f(0))+(L_a+L_L)i_a(0)(L_f s+R_f)}{(L_f s+R_f)((L_a+L_L)s+R_a+R_L)}$$
Note that when the system reaches its steady state condition, $\frac{di_f(t)}{dt} = 0$ and $\frac{di_a(t)}{dt} = 0$, from which the following equations are obtained for steady state operation:
$$V_f = R_f i_f(\infty)$$
$$K_e \omega i_f(\infty) = (R_a + R_L)i_a(\infty)$$
that is,
$$i_f(\infty) = \frac{V_f}{R_f}$$
$$i_a(\infty) = \frac{K_e \omega i_f(\infty)}{R_a+R_L}$$
Example 3.2: A separately excited DC generator operating at 1500rpm has the following parameters: $R_a = 0.2\Omega$, $L_a = 2.5mH$, $R_f = 3\Omega$, $L_f = 25mH$, and $K_e = 0.191$. If a DC voltage of 120V is suddenly applied to the field winding under a load with $R_L = 40\Omega$ and $L_L = 40mH$, determine the field current, armature current, and generated voltage as a function of time, the approximate time to reach the steady-state condition, and the steady-state values of the field current and induced voltage.

Solution: The Laplace transform of the field voltage is $V_f(s) = \frac{120}{s}$. The initial conditions are $i_f(0) = 0$ and $i_a(0) = 0$. The field current in s-domain is given by
$$I_f(s) = \frac{V_f(s)+L_f i_f(0)}{L_f s+R_f} = \frac{\frac{120}{s}+0.025\times 0}{0.025s+3} = \frac{120}{s(0.025s+3)} = \frac{\frac{120}{0.025}}{s\left(s+\frac{3}{0.025}\right)} =$$
$$\frac{4800}{s(s+120)} = \frac{A}{s} + \frac{B}{s+120} = \frac{40}{s} + \frac{-40}{s+120}$$
where
$$A = s\frac{4800}{s(s+120)}\bigg|_{s=0} = \frac{4800}{0+120} = 40$$
$$B = (s+120)\frac{4800}{s(s+120)}\bigg|_{s=-120} = \frac{4800}{-120} = -40$$
Therefore, the field current in time domain is
$$i_f(t) = 40 - 40e^{-120t}$$

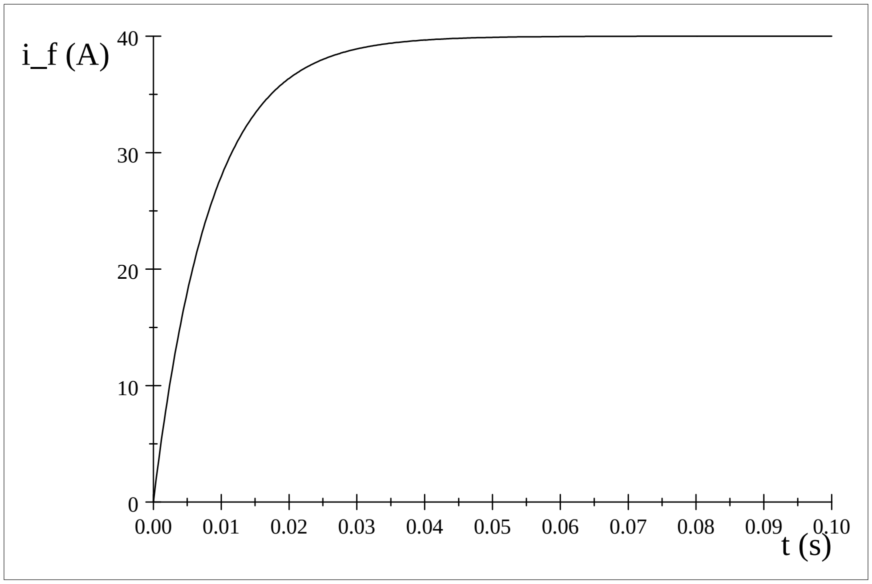

Figure3.4 The field current of the dc generator

The generator speed $\omega = \frac{2\pi n}{60} = \frac{2\pi\times1500}{60} = 157 rad/s$. The armature current in s-domain is

$$
\begin{aligned}
I_a(s) &= \frac{K_e\omega(V_f(s) + L_f i_f(0)) + (L_a + L_L)i_a(0)(L_f s + R_f)}{(L_f s + R_f)((L_a + L_L)s + R_a + R_L)} \\
&= \frac{0.191 \times 157 \times (\frac{120}{s} + 0.025 \times 0) + (0.0025 + 0.04) \times 0 \times (0.025s + 3)}{(0.025s + 3)((0.0025 + 0.04)s + 0.2 + 40)} \\
&= \frac{0.191 \times 157 \times 120}{s(0.025s + 3)(0.0425s + 40.2)} \\
&= \frac{\frac{0.191\times157\times120}{0.025\times0.0425}}{s\left(s + \frac{3}{0.025}\right)\left(s + \frac{40.2}{0.0425}\right)} \\
&= \frac{3.3868 \times 10^6}{s(s + 120.0)(s + 945.88)} \\
&= \frac{A}{s} + \frac{B}{s + 120} + \frac{C}{s + 945.88} \\
&= \frac{29.838}{s} + \frac{-34.174}{s + 120} + \frac{4.3355}{s + 945.88}
\end{aligned}
$$

where

$$A = s\frac{3.3868\times10^6}{s(s+120.0)(s+945.88)}\bigg|_{s=0} = \frac{3.3868\times10^6}{(0+120)(0+945.88)} = 29.838$$

$$B = (s + 120)\frac{3.3868\times10^6}{s(s+120.0)(s+945.88)}\bigg|_{s=-120} = \frac{3.3868\times10^6}{(-120)(-120+945.88)} = -34.174$$

$$C = (s + 945.88)\frac{3.3868\times10^6}{s(s+120.0)(s+945.88)}\bigg|_{s=-945.88} = \frac{3.3868\times10^6}{(-945.88)(-945.88+120)} = 4.3355$$

Therefore, the field current in time domain is

$$i_a(t) = 29.838 - 34.174e^{-120t} + 4.3355e^{-945.88t}$$

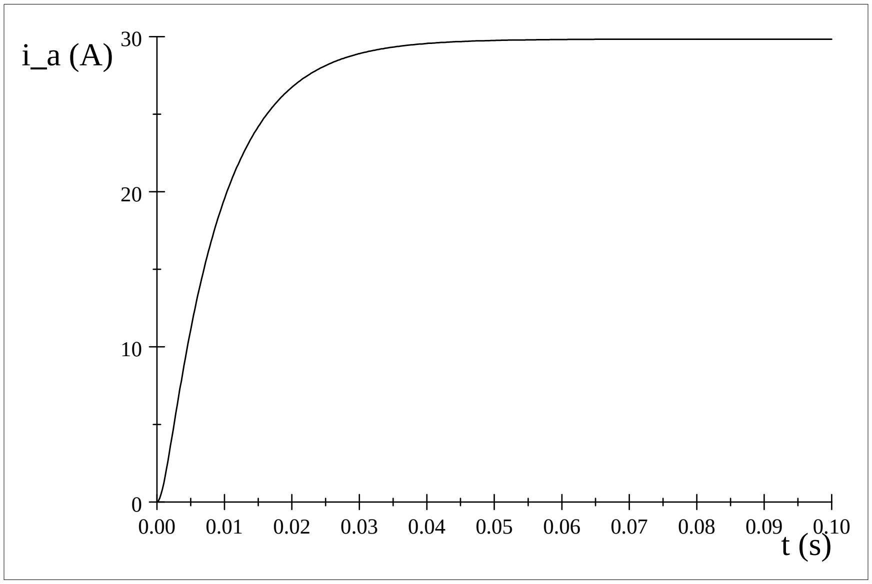

Figure3.5 The armature current of the dc generator

The induced voltage is given by

$$e_a(t) = K_e\omega i_f(t) = 0.191 \times 157 \times (40 - 40e^{-120t}) = 1199.5 - 1199.5e^{-120t}$$

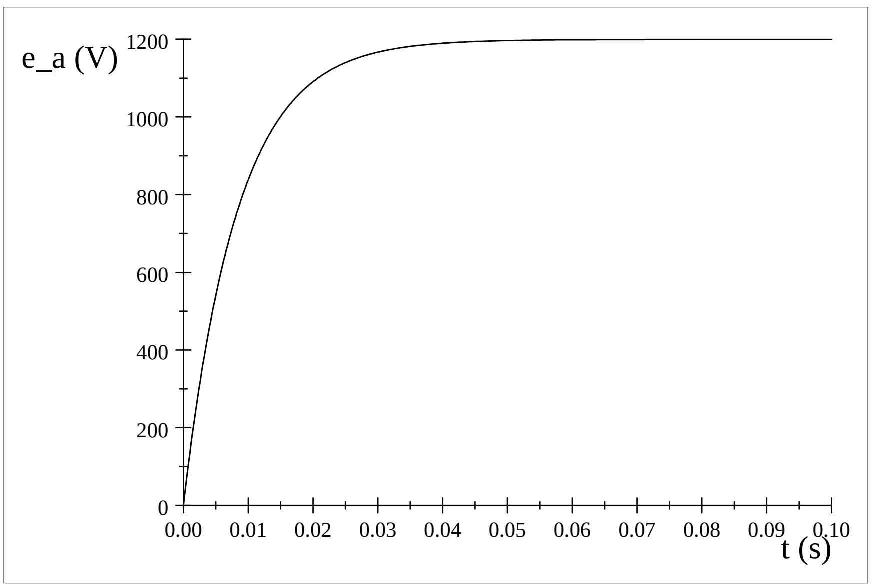

Figure3.6 The induced voltage of the dc generator

For practical purposes, the field current reaches its steady-state value after five time constant $5\tau_f = 5\frac{L_f}{R_f} = 5\frac{0.025}{3} = 0.042s$.

The final values of the field current, armature current, and induced voltage are $i_f(\infty) = 40A = \frac{V_f}{R_f}$, $i_a(\infty) = 29.838A = \frac{K_e\omega i_f(\infty)}{R_a+R_L}$, and $e_a(\infty) = 1199.5V = K_e\omega i_f(\infty)$.

3.3 DC Motor Dynamics

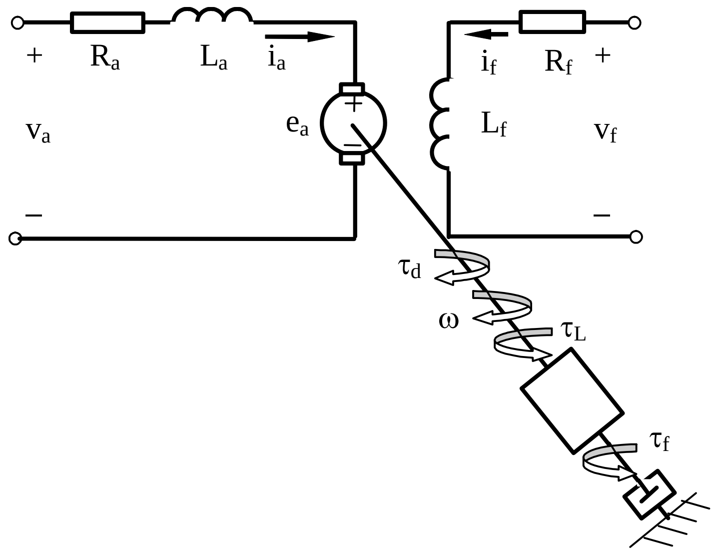

Figure3.7 DC motor dynamics

A DC motor is mainly composed of a stator, rotor, and commutator. The field winding is placed on the stator, which is also called the stator winding while the armature winding is mounted on the rotor, which is also referred to as the rotor winding. A pulsating induced voltage in the armature winding is converted to a DC voltage through the commutator. The equivalent circuit for a separately excited DC motor, together with a mechanical load, is shown in Figure 3.7.

For the field circuit, it follows from KVL that

$$v_f(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt}$$

where v_f, i_f, R_f, and L_f are the field voltage, current, resistance, and inductance, respectively.

For the armature circuit, according to KVL, we obtain

$$v_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + e_a(t)$$

where v_a, i_a, R_a, and L_a are the armature voltage, current, resistance, and inductance, respectively, and e_a is the back emf, which is determined by

$$e_a(t) = K_e i_f(t)\omega(t)$$

where K_e is the voltage constant and $\omega(t)$ is the angular speed of the motor.

For the mechanical load, it follows from Newton's law that

$$\tau_d(t) - \tau_L(t) - D\omega(t) = J\frac{d\omega(t)}{dt}$$

where D and J are the viscous friction coefficient and the moment of inertia of the rotating members, respectively, τ_L is the load torque and τ_d is the developed torque of the DC motor, which is determined by

$$\tau_d(t) = K_\tau i_f(t) i_a(t)$$

where K_τ is the torque constant, which is the same as the voltage constant K_e.

Substituting for e_a and τ_d in the three differential equations and solving them for the derivatives, it follows that

$$\frac{di_f(t)}{dt} = -\frac{R_f}{L_f} i_f(t) + \frac{1}{L_f} v_f(t)$$

$$\frac{di_a(t)}{dt} = -\frac{R_a}{L_a}i_a(t) - \frac{K_e}{L_a}i_f(t)\omega(t) + \frac{1}{L_a}v_a(t)$$
$$\frac{d\omega(t)}{dt} = \frac{K_e}{J}i_f(t)i_a(t) - \frac{1}{J}\tau_L(t) - \frac{D}{J}\omega(t)$$

which is a set of nonlinear differential equations. In these equations, both $v_f(t)$ and $v_a(t)$ can be adjusted to control the speed $\omega(t)$. When $v_f(t)$ is kept constant, that is, $i_f(t)$ is constant, the motor speed can be controlled by adjusting the armature voltage $v_a(t)$ and the motor is called the armature-controlled DC motor. On the other hand, when $v_a(t)$ is kept constant, the motor speed can be controlled by adjusting the field voltage $v_f(t)$ and the motor is called the field-controlled DC motor.

After the motor reaches the steady-state condition, $i_f(t)$, $i_a(t)$, and $\omega(t)$ remain constant, which implies that

$$\frac{di_f(t)}{dt} = 0, \frac{di_a(t)}{dt} = 0, \frac{d\omega(t)}{dt} = 0$$

Then, the following equations are obtained for the motor under steady-state condition.

$$v_f(\infty) = R_f i_f(\infty)$$
$$v_a(\infty) = R_a i_a(\infty) + e_a(\infty)$$
$$\tau_d(\infty) - \tau_L(\infty) - D\omega(\infty) = 0$$
$$e_a(\infty) = K_e i_f(\infty)\omega(\infty)$$
$$\tau_d(\infty) = K_e i_f(\infty)i_a(\infty)$$

or

$$0 = -R_f i_f(\infty) + v_f(\infty)$$
$$0 = -R_a i_a(\infty) - K_e i_f(\infty)\omega(\infty) + v_a(\infty)$$
$$0 = K_e i_f(\infty)i_a(\infty) - \tau_L(\infty) - D\omega(\infty)$$

from which one can determine the quantities $i_f(\infty)$, $i_a(\infty)$, and $\omega(\infty)$ under steady-state condition.

Example 3.3: A $240V$, $12hp$, separately excited DC motor has the following parameters $R_a = 0.28\Omega$, $L_a = 2.81mH$, $R_f = 320\Omega$, $L_f = 2H$, $K_e = 1.03$. $J = 0.087kg \cdot m$, and $D = 0.02N \cdot m \cdot s$. It is operating on a load of $15N \cdot m$ in the linear region of its magnetization characteristic. Determine the speed, field current, and armature current under steady-state condition.

Solution: The equations for the motor under steady-state condition are

$$0 = -R_f i_f(\infty) + v_f(\infty)$$
$$0 = -R_a i_a(\infty) - K_e i_f(\infty)\omega(\infty) + v_a(\infty)$$
$$0 = K_e i_f(\infty)i_a(\infty) - \tau_L(\infty) - D\omega(\infty)$$

Solving the first equation for $i_f(t)$ gives

$$i_f(\infty) = \frac{v_f(\infty)}{R_f} = \frac{240}{320} = 0.75A$$

Solving the second equation for $i_a(t)$ yields

$$i_a(\infty) = \frac{v_a(\infty)}{R_a} - \frac{K_e i_f(\infty)}{R_a}\omega(\infty)$$

Substituting this into the third equation produces

$$0 = K_e i_f(\infty)\left(\frac{v_a(\infty)}{R_a} - \frac{K_e i_f(\infty)}{R_a}\omega(\infty)\right) - \tau_L(\infty) - D\omega(\infty)$$

that is,

$$0 = K_e i_f(\infty)v_a(\infty) - K_e^2 i_f^2(\infty)\omega(\infty) - \tau_L(\infty)R_a - DR_a\omega(\infty)$$

Solving this for $\omega(\infty)$, we have

$$\omega(\infty) = \frac{K_e i_f(\infty) v_a(\infty) - \tau_L(\infty) R_a}{K_e^2 i_f^2(\infty) + D R_a} = \frac{1.03\times0.75\times240 - 15\times0.28}{1.03^2\times0.75^2 + 0.02\times0.28} = 300.82 rad/s$$

Therefore, the armature current is

$$i_a(\infty) = \frac{v_a(\infty)}{R_a} - \frac{K_e i_f(\infty)}{R_a}\omega(\infty) = \frac{240}{0.28} - \frac{1.03\times0.75}{0.28} \times 300.82 = 27.202A$$

3.4 Armature-Controlled DC Motors

For armature controlled DC motors, the field voltage is kept constant at V_f, so the field current is constant too, which implies that $\frac{di_f(t)}{dt} = 0$ and $i_f(t) = I_f = \frac{V_f}{R_f}$. The dynamic model for an armature controlled DC motor becomes

$$\frac{di_a(t)}{dt} = -\frac{R_a}{L_a} i_a(t) - \frac{K_e}{L_a} I_f \omega(t) + \frac{1}{L_a} v_a(t)$$

$$\frac{d\omega(t)}{dt} = \frac{K_e}{J} I_f i_a(t) - \frac{1}{J}\tau_L(t) - \frac{D}{J}\omega(t)$$

which is in the state-space form with state variables $i_a(t)$ and $\omega(t)$.

Taking the Laplace transform, together with initial conditions $i_a(0)$ and $\omega(0)$, gives

$$sI_a(s) - i_a(0) = -\frac{R_a}{L_a} I_a(s) - \frac{K_e}{L_a} I_f \omega(s) + \frac{1}{L_a} V_a(s)$$

$$s\omega(s) - \omega(0) = \frac{K_e}{J} I_f I_a(s) - \frac{1}{J}\tau_L(s) - \frac{D}{J}\omega(s)$$

or

$$L_a s I_a(s) - L_a i_a(0) = -R_a I_a(s) - K_e I_f \omega(s) + V_a(s)$$

$$J s\omega(s) - J\omega(0) = K_e I_f I_a(s) - \tau_L(s) - D\omega(s)$$

Solving the first equation for $I_a(s)$ yields

$$I_a(s) = \frac{(V_a(s) + L_a i_a(0)) - K_e I_f \omega(s)}{L_a s + R_a}$$

Substituting into the second equation gives

$$J s\omega(s) - J\omega(0) = K_e I_f \frac{(V_a(s) + L_a i_a(0)) - K_e I_f \omega(s)}{L_a s + R_a} - \tau_L(s) - D\omega(s)$$

that is,

$$(Js + D)\omega(s) = \frac{K_e I_f (V_a(s) + L_a i_a(0))}{L_a s + R_a} - \frac{(K_e I_f)^2 \omega(s)}{L_a s + R_a} + J\omega(0) - \tau_L(s)$$

Solving this for $\omega(s)$ produces

$$\omega(s) = \frac{K_e I_f (V_a(s) + L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js + D)(L_a s + R_a) + (K_e I_f)^2}$$

Then, the armature current is given by

$$
\begin{aligned}
I_a(s) &= \frac{(V_a(s)+L_a i_a(0)) - K_e I_f \omega(s)}{L_a s + R_a} \\
&= \frac{(V_a(s)+L_a i_a(0)) - K_e I_f \frac{K_e I_f (V_a(s)+L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js+D)(L_a s + R_a) + (K_e I_f)^2}}{L_a s + R_a} \\
&= \frac{\frac{(V_a(s)+L_a i_a(0))\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right)}{(Js+D)(L_a s+R_a)+(K_e I_f)^2} - K_e I_f \frac{K_e I_f (V_a(s)+L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js+D)(L_a s + R_a) + (K_e I_f)^2}}{L_a s + R_a} \\
&= \frac{\begin{array}{c}(V_a(s)+L_a i_a(0))\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right) \\ -K_e I_f (K_e I_f (V_a(s)+L_a i_a(0)) + (L_a s+R_a)(J\omega(0)-\tau_L(s)))\end{array}}{(L_a s+R_a)\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right)} \\
&= \frac{\begin{array}{c}(V_a(s)+L_a i_a(0))(Js+D)(L_a s+R_a) + (K_e I_f)^2 (V_a(s)+L_a i_a(0)) \\ -(K_e I_f)^2 (V_a(s)+L_a i_a(0)) - K_e I_f (L_a s+R_a)(J\omega(0)-\tau_L(s))\end{array}}{(L_a s+R_a)\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right)} \\
&= \frac{(V_a(s)+L_a i_a(0))(Js+D)(L_a s+R_a) - K_e I_f (L_a s+R_a)(J\omega(0)-\tau_L(s))}{(L_a s+R_a)\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right)} \\
&= \frac{(V_a(s)+L_a i_a(0))(Js+D) - K_e I_f (J\omega(0)-\tau_L(s))}{(Js+D)(L_a s+R_a)+(K_e I_f)^2}
\end{aligned}
$$

Example 3.4: (see Example 3.3) A $240V$, $12hp$, separately excited DC motor has the following parameters $R_a = 0.28\Omega$, $L_a = 2.81mH$, $R_f = 320\Omega$, $L_f = 2H$, $K_e = 1.03$. $J = 0.087kg \cdot m$, and $D = 0.02N \cdot m \cdot s$. Determine its speed and armature current as a function of time when it is suddenly connected to a 240V DC source at no load condition.

Solution: Prior to the application of armature voltage the motor speed and armature current are zero. That is, at $t = 0$, $i_a(0) = 0$ and $\omega(0) = 0$. In addition, the load torque is zero because the motor operates at no load. That is, $\tau_L(t) = 0$. The field current is

$$I_f = \frac{V_f}{R_f} = \frac{240}{320} = 0.75A$$

Note that the armature voltage $v_a(t)$ is a step signal with amplitude of 240V, so its Laplace transform is $V_a(s) = \frac{240}{s}$.

Therefore, we have

$$
\begin{aligned}
\omega(s) &= \frac{K_eI_f(V_a(s)+L_ai_a(0))+(L_as+R_a)(J\omega(0)-\tau_L(s))}{(Js+D)(L_as+R_a)+(K_eI_f)^2} \\
&= \frac{1.03\times 0.75\times\frac{240}{s}}{(0.087s+0.02)(0.00281s+0.28)+(1.03\times 0.75)^2} \\
&= \frac{185.4}{s(2.4447\times 10^{-4}s^2+2.4416\times 10^{-2}s+0.60236)} \\
&= \frac{\frac{185.4}{2.4447\times 10^{-4}}}{s\left(s^2+\frac{2.4416\times 10^{-2}}{2.4447\times 10^{-4}}s+\frac{0.60236}{2.4447\times 10^{-4}}\right)} \\
&= \frac{7.5838\times 10^5}{s(s^2+99.873s+2463.9)} \\
&= \frac{7.5838\times 10^5}{s(s+44.482)(s+55.391)}
\end{aligned}
$$

In order to determine the inverse Laplace transform, we expand $\omega(s)$ into partial fractions as

$$\omega(s) = \frac{A}{s}+\frac{B}{s+44.482}+\frac{C}{s+55.391}$$

where A, B, and C can now be determined by the root-substitution method. Thus,

$$A = s\frac{7.5838\times 10^5}{s(s+44.482)(s+55.391)}\bigg|_{s=0} = \frac{7.5838\times 10^5}{(0+44.482)(0+55.391)} = 307.80$$

$$B = (s+44.482)\frac{7.5838\times 10^5}{s(s+44.482)(s+55.391)}\bigg|_{s=-44.482} = \frac{7.5838\times 10^5}{(-44.482)(-44.482+55.391)} = -1562.9$$

$$C = (s+55.391)\frac{7.5838\times 10^5}{s(s+44.482)(s+55.391)}\bigg|_{s=-55.391} = \frac{7.5838\times 10^5}{(-55.391)(-55.391+44.482)} = 1255.1$$

Finally, we can take the inverse Laplace transform of

$$\omega(s) = \frac{307.80}{s}+\frac{-1562.9}{s+44.482}+\frac{1255.1}{s+55.391}$$

and get the angular velocity as

$$\omega(t) = 307.80-1562.9e^{-44.482t}+1255.1e^{-55.391t}$$

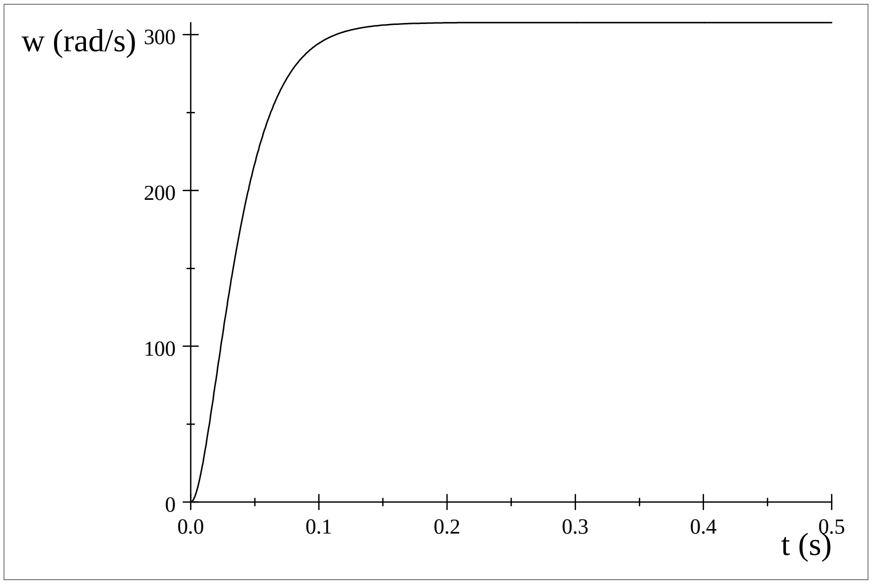

Figure3.8 The motor speed

The Laplace transform of the armature current is

$$
\begin{aligned}
I_a(s) &= \frac{(V_a(s) + L_a i_a(0))(Js + D) - K_e I_f (J\omega(0) - \tau_L(s))}{(Js + D)(L_a s + R_a) + (K_e I_f)^2} \\
&= \frac{\frac{240}{s}(0.087s + 0.02)}{(0.087s + 0.02)(0.00281s + 0.28) + (1.03 \times 0.75)^2} \\
&= \frac{240(0.087s + 0.02)}{s(2.4447 \times 10^{-4} s^2 + 2.4416 \times 10^{-2} s + 0.60236)} \\
&= \frac{\frac{240 \times 0.087}{2.4447 \times 10^{-4}} s + \frac{240 \times 0.02}{2.4447 \times 10^{-4}}}{s\left(s^2 + \frac{2.4416 \times 10^{-2}}{2.4447 \times 10^{-4}} s + \frac{0.60236}{2.4447 \times 10^{-4}}\right)} \\
&= \frac{85409s + 19634}{s(s^2 + 99.873s + 2463.9)} \\
&= \frac{85409s + 19634}{s(s + 44.482)(s + 55.391)}
\end{aligned}
$$

In terms of its partial fraction expansion, $I_a(s)$ can be written as

$$i_a(s) = \frac{A}{s} + \frac{B}{s+44.482} + \frac{C}{s+55.391} = \frac{7.9687}{s} + \frac{7788.8}{s+44.482} + \frac{-7796.7}{s+55.391}$$

where

$$A = s\frac{85409s+19634}{s(s+44.482)(s+55.391)}\bigg|_{s=0} = \frac{85409 \times 0 + 19634}{(0+44.482)(0+55.391)} = 7.9687$$

$$B = (s + 44.482)\frac{85409s+19634}{s(s+44.482)(s+55.391)}\bigg|_{s=-44.482} = \frac{85409 \times (-44.482) + 19634}{(-44.482)(-44.482+55.391)} = 7788.8$$

$$C = (s + 55.391)\frac{85409s+19634}{s(s+44.482)(s+55.391)}\bigg|_{s=-55.391} = \frac{85409 \times (-55.391) + 19634}{(-55.391)(-55.391+44.482)} = -7796.7$$

Finally, we obtain the armature current as

$$i_a(t) = 7.9687 + 7788.8e^{-44.482t} - 7796.7e^{-55.391t}$$

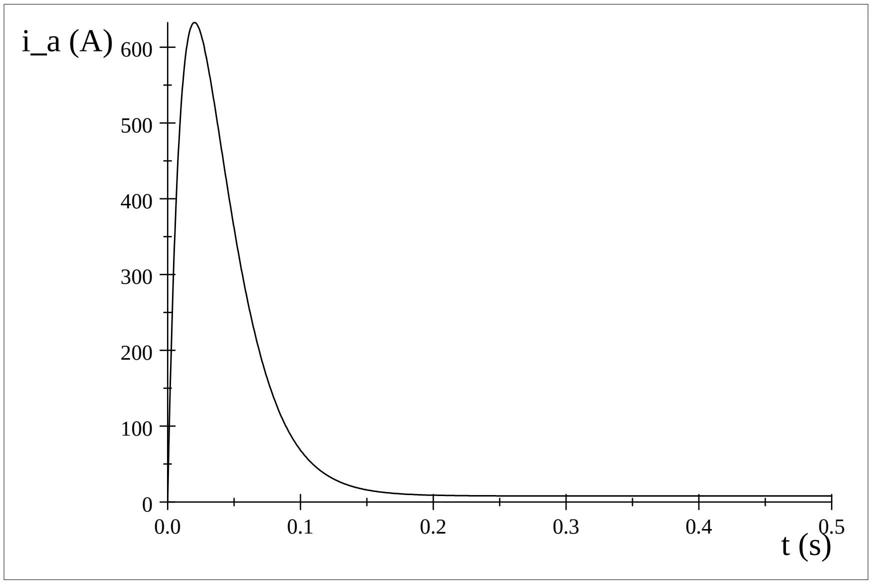

Figure3.9 The motor armature current

The rated current is $I_a(rating) = \frac{12\times746}{240} = 37.3A$. The starting current is way too high so that the motor will be burnt.

Note that the mechanical time constant is $\tau_m = \frac{J}{D} = \frac{0.087}{0.02} = 4.35s$ and the electrical time constant is

$\tau_e = \frac{L_a}{R_a} = \frac{0.00281}{0.28} = 1.0036\times10^{-2}s$.

Example 3.5: (See Example 3.3) A $240V$, $12hp$, separately excited DC motor has the following parameters $R_a = 0.28\Omega$, $L_a = 2.81mH$, $R_f = 320\Omega$, $L_f = 2H$, $K_e = 1.03$. $J = 0.087kg\cdot m$, and $D = 0.02N\cdot m\cdot s$. Determine its speed and armature current as a function of time when it is suddenly connected to a $30V$ DC source at a load of $15N\cdot m$.

Solution: Prior to the application of armature voltage the motor speed and armature current are zero. That is, at $t = 0$, $i_a(0) = 0$ and $\omega(0) = 0$. In addition, the load torque is $15N\cdot m$, that is, $\tau_L(t) = 15$. The field current is

$$I_f = \frac{V_f}{R_f} = \frac{240}{320} = 0.75A$$

Note that $V_a(s) = \frac{30}{s}$ and $\tau_L(s) = \frac{15}{s}$.

Therefore, we have

$$
\begin{aligned}
\omega(s) &= \frac{K_e I_f (V_a(s) + L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js + D)(L_a s + R_a) + (K_e I_f)^2} \\
&= \frac{1.03 \times 0.75 \times \frac{30}{s} + (0.00281s + 0.28)(-\frac{15}{s})}{(0.087s + 0.02)(0.00281s + 0.28) + (1.03 \times 0.75)^2} \\
&= \frac{1.03 \times 0.75 \times 30 + (0.00281s + 0.28)(-15)}{s(2.4447 \times 10^{-4} s^2 + 2.4416 \times 10^{-2} s + 0.60236)} \\
&= \frac{18.975 - 0.04215s}{s(2.4447 \times 10^{-4} s^2 + 2.4416 \times 10^{-2} s + 0.60236)} \\
&= \frac{\frac{18.975}{2.4447\times10^{-4}} - \frac{0.04215}{2.4447\times10^{-4}} s}{s\left(s^2 + \frac{2.4416\times10^{-2}}{2.4447\times10^{-4}} s + \frac{0.60236}{2.4447\times10^{-4}}\right)} \\
&= \frac{77617 - 172.41s}{s(s^2 + 99.873s + 2463.9)} \\
&= \frac{77617 - 172.41s}{s(s + 44.482)(s + 55.391)}
\end{aligned}
$$

In order to determine the inverse Laplace transform, we expand $\omega(s)$ into partial fractions as

$$\omega(s) = \frac{A}{s} + \frac{B}{s+44.482} + \frac{C}{s+55.391}$$

where A, B, and C can now be determined by the root-substitution method. Thus,

$$
\begin{aligned}
A &= s\frac{77617-172.41s}{s(s+44.482)(s+55.391)}\bigg|_{s=0} = \frac{77617-172.41\times0}{(0+44.482)(0+55.391)} = 31.502 \\
B &= (s + 44.482)\frac{77617-172.41s}{s(s+44.482)(s+55.391)}\bigg|_{s=-44.482} = \frac{77617-172.41\times(-44.482)}{(-44.482)(-44.482+55.391)} = -175.76 \\
C &= (s + 55.391)\frac{77617-172.41s}{s(s+44.482)(s+55.391)}\bigg|_{s=-55.391} = \frac{77617-172.41\times(-55.391)}{(-55.391+44.482)(-55.391)} = 144.25
\end{aligned}
$$

Finally, we can take the inverse Laplace transform of

$$\omega(s) = \frac{31.502}{s} + \frac{-175.76}{s+44.482} + \frac{144.25}{s+55.391}$$

and get the angular velocity as

$$\omega(t) = 31.502 - 175.76e^{-44.482t} + 144.25e^{-55.391t}$$

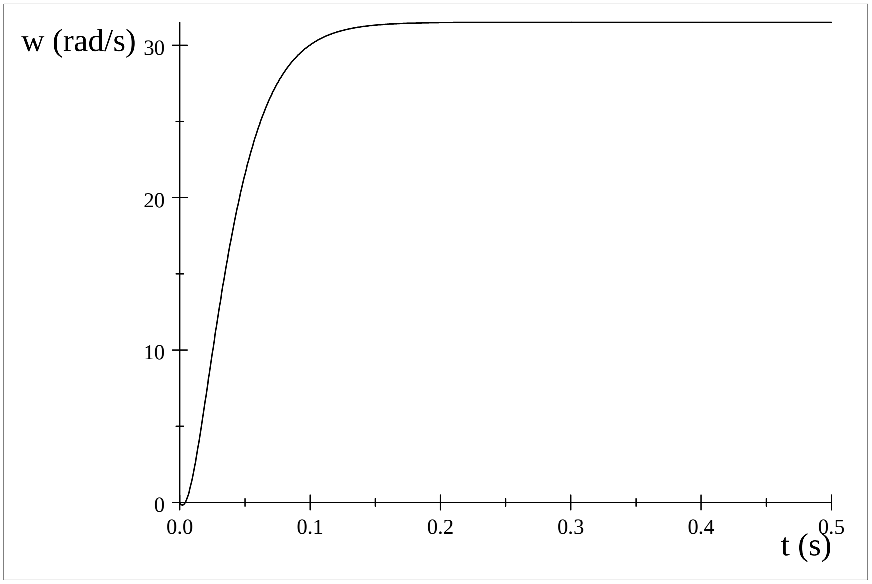

Figure3.10 The motor speed

The Laplace transform of the armature current is

$$\begin{aligned}
I_a(s) &= \frac{(V_a(s)+L_a i_a(0))(Js+D) - K_e I_f(J\omega(0) - \tau_L(s))}{(Js+D)(L_a s+R_a)+(K_e I_f)^2} \\
&= \frac{\frac{30}{s}(0.087s+0.02) - 1.03\times 0.75(-\frac{15}{s})}{(0.087s+0.02)(0.00281s+0.28)+(1.03\times 0.75)^2} \\
&= \frac{30(0.087s+0.02) - 1.03\times 0.75(-15)}{s(2.4447\times 10^{-4}s^2 + 2.4416\times 10^{-2}s + 0.60236)} \\
&= \frac{2.61s+12.188}{s(2.4447\times 10^{-4}s^2 + 2.4416\times 10^{-2}s + 0.60236)} \\
&= \frac{\frac{2.61}{2.4447\times 10^{-4}}s + \frac{12.188}{2.4447\times 10^{-4}}}{s\left(s^2 + \frac{2.4416\times 10^{-2}}{2.4447\times 10^{-4}}s + \frac{0.60236}{2.4447\times 10^{-4}}\right)} \\
&= \frac{10676s+49855}{s(s^2+99.873s+2463.9)} \\
&= \frac{10676s+49855}{s(s+44.482)(s+55.391)}
\end{aligned}$$

In terms of its partial fraction expansion, $I_a(s)$ can be written as

$$\omega(s) = \frac{A}{s} + \frac{B}{s+44.482} + \frac{C}{s+55.391} = \frac{20.234}{s} + \frac{875.9}{s+44.482} + \frac{-896.14}{s+55.391}$$

where

$$A = s\frac{10676s+49855}{s(s+44.482)(s+55.391)}\Bigg|_{s=0} = \frac{10676\times 0+49855}{(0+44.482)(0+55.391)} = 20.234$$

$$B = (s+44.482)\frac{10676s+49855}{s(s+44.482)(s+55.391)}\Bigg|_{s=-44.482} = \frac{10676\times(-44.482)+49855}{(-44.482)(-44.482+55.391)} = 875.9$$

$$C = (s+55.391)\frac{10676s+49855}{s(s+44.482)(s+55.391)}\Bigg|_{s=-55.391} = \frac{10676\times(-55.391)+49855}{(-55.391+44.482)(-55.391)} = -896.14$$

Finally, we obtain the armature current as

$i_a(t) = 20.234 + 875.9e^{-44.482t} - 896.14e^{-55.391t}$

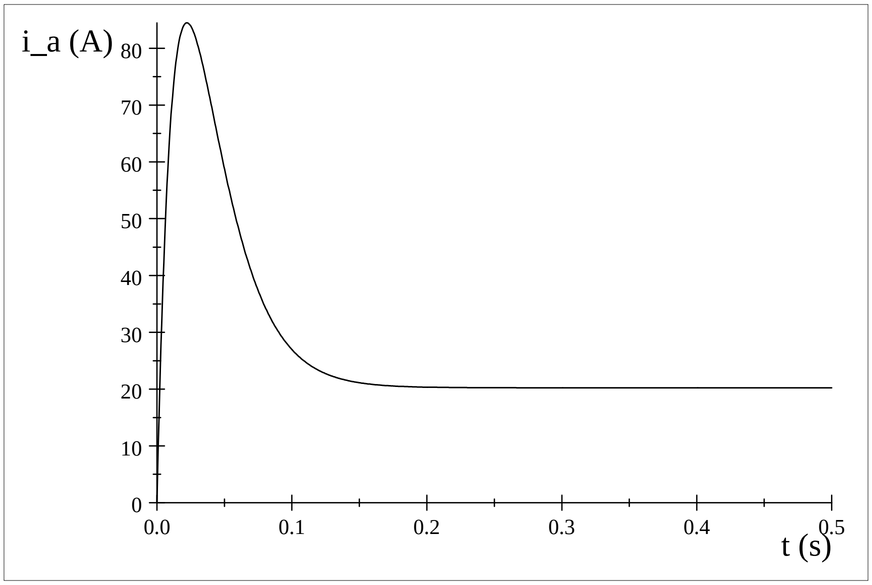

Figure3.11 The motor armature current

3.5 Field-Controlled DC Motors

In an armature-controlled DC motor, the field current is kept at a constant level and the armature voltage is adjusted to vary the speed below its rated speed. In a field-controlled DC motor, however, we will change the field current in order to obtain a motor speed higher than its rated speed.

The mathematical model for a field-controlled DC motor is given below.

$$\frac{di_f(t)}{dt} = -\frac{R_f}{L_f} i_f(t) + \frac{1}{L_f} v_f(t)$$
$$\frac{di_a(t)}{dt} = -\frac{R_a}{L_a} i_a(t) - \frac{K_e}{L_a} i_f(t)\omega(t) + \frac{V_a}{L_a}$$
$$\frac{d\omega(t)}{dt} = \frac{K_e}{J} i_f(t) i_a(t) - \frac{1}{J}\tau_L(t) - \frac{D}{J}\omega(t)$$

It is clear that these equations are nonlinear because of the products of the state variables in these equations. As a result, the Laplace transform approach would not be appropriate to get closed-form solutions for $i_f(t)$, $i_a(t)$ and $\omega(t)$. However, a simplifying assumption can be made to linearize these equations.

In an electric motor, the time constant of the electric circuit is much smaller than the time constant of the mechanical parts. Therefore, it can be considered that the time constant of the field circuit is much smaller than the mechanical time constant of the motor. The field current reaches its steady-state before the armature responds to the changes in the field current. Therefore, we have

$$\frac{di_f(t)}{dt} = -\frac{R_f}{L_f} i_f(t) + \frac{1}{L_f} v_f(t)$$
$$\frac{di_a(t)}{dt} = -\frac{R_a}{L_a} i_a(t) - \frac{K_e}{L_a} I_f\omega(t) + \frac{V_a}{L_a}$$
$$\frac{d\omega(t)}{dt} = \frac{K_e}{J} I_f i_a(t) - \frac{1}{J}\tau_L(t) - \frac{D}{J}\omega(t)$$

Taking the Laplace transform gives

$$sI_f(s) - i_f(0) = -\frac{R_f}{L_f}I_f(s) + \frac{1}{L_f}V_f(s)$$
$$sI_a(s) - i_a(0) = -\frac{R_a}{L_a}I_a(s) - \frac{K_e}{L_a}I_f\omega(s) + \frac{1}{L_a}V_a(s)$$
$$s\omega(s) - \omega(0) = \frac{K_e}{J}I_fI_a(s) - \frac{1}{J}\tau_L(s) - \frac{D}{J}\omega(s)$$

or

$$L_fsI_f(s) - L_fi_f(0) = -R_fI_f(s) + V_f(s)$$
$$L_asI_a(s) - L_ai_a(0) = -R_aI_a(s) - K_eI_f\omega(s) + V_a(s)$$
$$Js\omega(s) - J\omega(0) = K_eI_fI_a(s) - \tau_L(s) - D\omega(s)$$

Solving these equations yields

$$I_f(s) = \frac{V_f(s)+L_fi_f(0)}{L_fs+R_f}$$
$$\omega(s) = \frac{K_eI_f(V_a(s)+L_ai_a(0))+(L_as+R_a)(J\omega(0)-\tau_L(s))}{(Js+D)(L_as+R_a)+(K_eI_f)^2}$$
$$I_a(s) = \frac{(V_a(s)+L_ai_a(0))(Js+D)-K_eI_f(J\omega(0)-\tau_L(s))}{(Js+D)(L_as+R_a)+(K_eI_f)^2}$$

Example 3.6: (See Example 3.3) A $240V$, $12hp$, separately excited DC motor has the following parameters $R_a = 0.28\Omega$, $L_a = 2.81mH$, $R_f = 320\Omega$, $L_f = 2H$, $K_e = 1.03$. $J = 0.087kg \cdot m$, and $D = 0.02N \cdot m \cdot s$. It is operating on a load of $15N \cdot m$ in the linear region of its magnetization characteristic. Determine its speed, field current, and armature current as a function of time when the field voltage is suddenly reduced from $240V$ to $192V$ at $t = 0$.

Solution: Since the motor has already been operating at steady state on a load of $\tau_L = 15N \cdot m$ before the field voltage is suddenly changed, we have to evaluate the initial conditions on $i_f(t),\ i_a(t)$ and $\omega(t)$ from the equations for the steady-state operation, which is done in Example 3.2 and the initial values are

$$i_f(0) = 0.75A, \omega(0) = 300.82rad/s, i_a(0) = 27.202A$$

First, we will determine the field current as follows:

$$I_f(s) = \frac{V_f(s)+L_fi_f(0)}{L_fs+R_f} = \frac{V_f(s)+L_fi_f(0)}{L_fs+R_f} = \frac{\frac{192}{s}+2\times0.75}{2s+320} = \frac{96+0.75s}{s(s+160)} = \frac{A}{s} + \frac{B}{s+160} = \frac{0.6}{s} + \frac{0.15}{s+160}$$

where

$$A = s\frac{96+0.75s}{s(s+160)}\bigg|_{s=0} = \frac{96+0.75\times0}{(0+160)} = 0.6$$
$$B = s\frac{96+0.75s}{s(s+160)}\bigg|_{s=-160} = \frac{96+0.75\times(-160)}{-160} = 0.15$$

Taking the inverse Laplace transform produces

$$i_f(t) = 0.6 + 0.15e^{-160t}$$

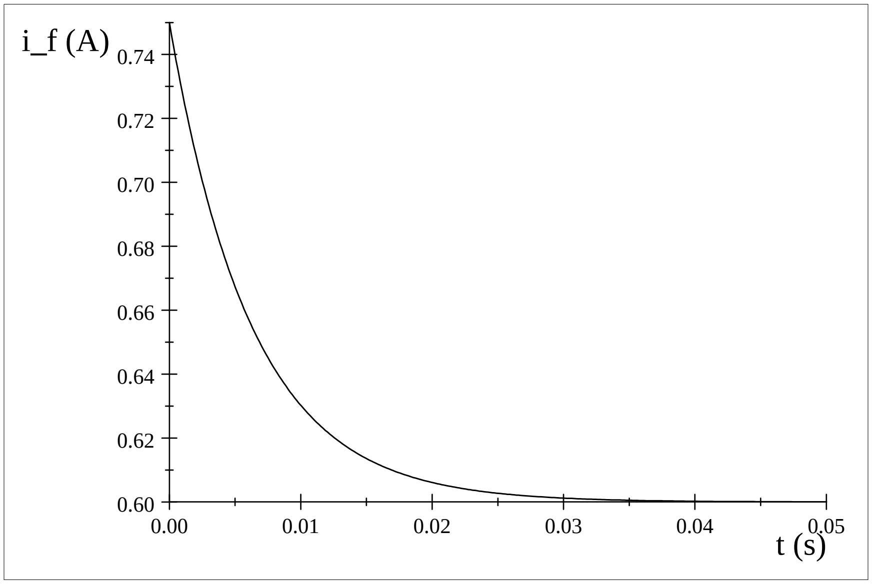

Figure3.12 The motor field current

which has a steady state value $I_f = 0.6A$, which is the same as $\frac{V_f}{R_f} = \frac{192}{320} = 0.6$.

For the motor speed, we have

$$
\begin{aligned}
\omega(s) &= \frac{K_e I_f(V_a(s) + L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js + D)(L_a s + R_a) + (K_e I_f)^2} \\
&= \frac{1.03 \times 0.6 \times (\frac{240}{s} + 0.00281 \times 27.2) + (0.00281s + 0.28)(0.087 \times 300.79 - \frac{15}{s})}{(0.087s + 0.02)(0.00281s + 0.28) + (1.03 \times 0.6)^2} \\
&= \frac{1.03 \times 0.6 \times (240 + 0.00281 \times 27.2s) + (0.00281s + 0.28)(0.087 \times 300.79s - 15)}{s(2.4447 \times 10^{-4}s^2 + 2.4416 \times 10^{-2}s + 0.38752)} \\
&= \frac{7.3534 \times 10^{-2}s^2 + 7.3323s + 144.12}{s(2.4447 \times 10^{-4}s^2 + 2.4416 \times 10^{-2}s + 0.38752)} \\
&= \frac{\frac{7.3534\times10^{-2}}{2.4447\times10^{-4}}s^2 + \frac{7.3323}{2.4447\times10^{-4}}s + \frac{144.12}{2.4447\times10^{-4}}}{s\left(s^2 + \frac{2.4416\times10^{-2}}{2.4447\times10^{-4}}s + \frac{0.38752}{2.4447\times10^{-4}}\right)} \\
&= \frac{300.79s^2 + 29993.s + 5.8952 \times 10^5}{s(s^2 + 99.873s + 1585.1)} \\
&= \frac{300.79s^2 + 29993.s + 5.8952 \times 10^5}{s(s + 19.794)(s + 80.079)} \\
&= \frac{A}{s} + \frac{B}{s + 19.794} + \frac{C}{s + 80.079} \\
&= \frac{371.92}{s} + \frac{-95.274}{s + 19.794} + \frac{24.147}{s + 80.079}
\end{aligned}
$$

where A, B, and C are determined by

$$A = s\frac{300.79s^2+29993s+5.8952\times10^5}{s(s+19.794)(s+80.079)}\bigg|_{s=0} = \frac{300.79\times0^2+29993\times0+5.8952\times10^5}{(0+19.794)(0+80.079)} = 371.92$$

$B = (s + 19.794)\frac{300.79s^2+29993s+5.8952\times10^5}{s(s+19.794)(s+80.079)}\Big|_{s=-19.794} = \frac{300.79\times(-19.794)^2+29993\times(-19.794)+5.8952\times10^5}{(-19.794)(-19.794+80.079)} = -95.274$

$C = (s + 80.079)\frac{300.79s^2+29993s+5.8952\times10^5}{s(s+19.794)(s+80.079)}\Big|_{s=-80.079} = \frac{300.79\times(-80.079)^2+29993\times(-80.079)+5.8952\times10^5}{(-80.079)(-80.079+19.794)} = 24.147$

Finally, we can take the inverse Laplace transform to get the angular velocity as

$$\omega(t) = 371.92 - 95.274e^{-19.794t} + 24.147e^{-80.079t}$$

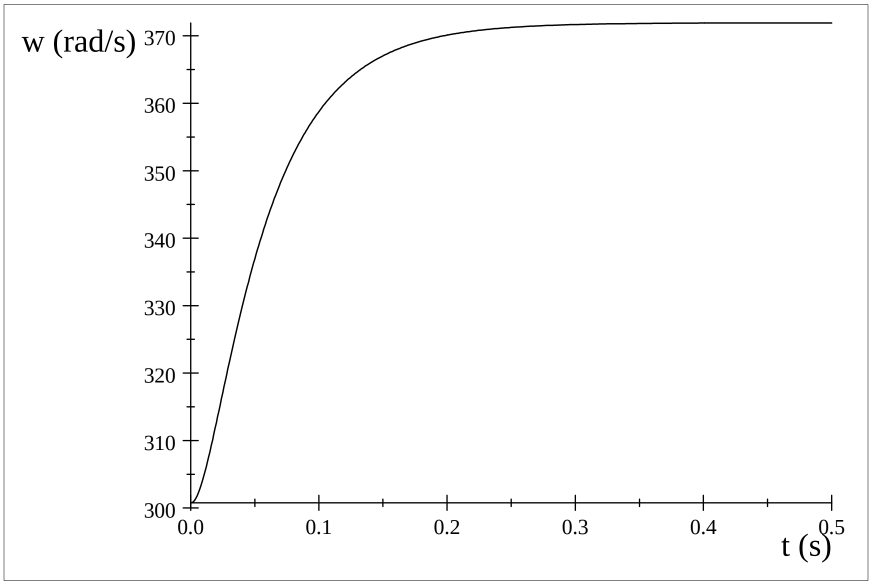

Figure3.13 The motor speed

The steady-state speed is $371.92rad/s$, which is higher than the steady-state speed $300rad/s$ (see Example 3.3) corresponding to the rated field current $0.75A$. This confirms that the speed is increased with a lower field current.

The Laplace transform of the armature current is

$$
\begin{aligned}
I_a(s) &= \frac{(V_a(s)+L_a i_a(0))(Js+D)-K_e I_f(J\omega(0)-\tau_L(s))}{(Js+D)(L_a s+R_a)+(K_e I_f)^2} \\
&= \frac{(\frac{240}{s}+0.00281\times 27.2)(0.087s+0.02)-1.03\times 0.6(0.087\times 300.79-\frac{15}{s})}{(0.087s+0.02)(0.00281s+0.28)+(1.03\times 0.6)^2} \\
&= \frac{(240+0.00281\times 27.2s)(0.087s+0.02)-1.03\times 0.6(0.087\times 300.79s-15)}{s(2.4447\times 10^{-4}s^2+2.4416\times 10^{-2}s+0.38752)} \\
&= \frac{6.6496\times 10^{-3}s^2+4.7093s+14.07}{s(2.4447\times 10^{-4}s^2+2.4416\times 10^{-2}s+0.38752)} \\
&= \frac{\frac{6.6496\times 10^{-3}}{2.4447\times 10^{-4}}s^2+\frac{4.7093}{2.4447\times 10^{-4}}s+\frac{14.07}{2.4447\times 10^{-4}}}{s\left(s^2+\frac{2.4416\times 10^{-2}}{2.4447\times 10^{-4}}s+\frac{0.38752}{2.4447\times 10^{-4}}\right)} \\
&= \frac{27.2s^2+19263s+57553}{s(s^2+99.873s+1585.1)} \\
&= \frac{27.2s^2+19263s+57553}{s(s+19.794)(s+80.079)} \\
&= \frac{A}{s}+\frac{B}{s+19.794}+\frac{C}{s+80.079} \\
&= \frac{36.309}{s}+\frac{262.37}{s+19.794}+\frac{-271.48}{s+80.079}
\end{aligned}
$$

where

$$A = s\frac{27.2s^2+19263s+57553}{s(s+19.794)(s+80.079)}\bigg|_{s=0} = \frac{27.2\times(0)^2+19263\times(0)+57553}{(0+19.794)(0+80.079)} = 36.309$$

$$B = (s+19.794)\frac{27.2s^2+19263s+57553}{s(s+19.794)(s+80.079)}\bigg|_{s=-19.794} = \frac{27.2\times(-19.794)^2+19263\times(-19.794)+57553}{(-19.794)(-19.794+80.079)} = 262.37$$

$$C = (s+80.079)\frac{27.2s^2+19263s+57553}{s(s+19.794)(s+80.079)}\bigg|_{s=-80.079} = \frac{27.2\times(-80.079)^2+19263\times(-80.079)+57553}{(-80.079)(-80.079+19.794)} = -271.48$$

Finally, we obtain the armature current as

$$i_a(t) = 36.309+262.37e^{-19.794t}-271.48e^{-80.079t}$$

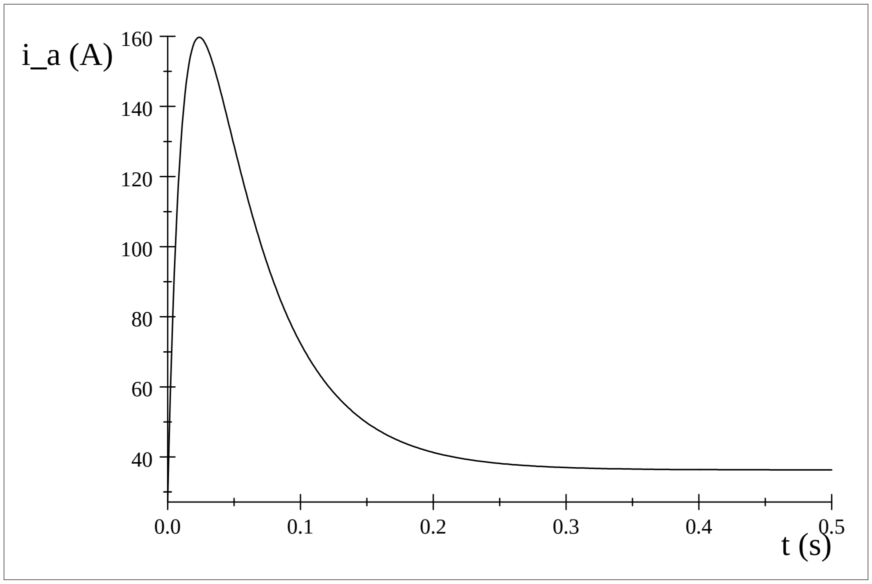

Figure3.14 The motor armature current

It is clear that the field current reaches its steady state at about 30ms whereas it takes about 300ms for the speed and thereby the armature current to do so. This is consistent with our assumption that the mechanical response is much slower than the changes in the field current.

It is important to note that the armature current reaches its peak at 160A, which is well over its rated value. This is mainly caused by the large mechanical time constant of the motor that does not allow a rapid change in the back emf of the motor. Therefore, it is recommended that the field current be gradually varied so that high currents will not take place in the armature circuit.

3.6 Voltage Control of DC Generators

3.7 Speed Control of DC Motors

Formula Sheet for the Midterm Test:

$\mathcal{F} = Ni$

$\vec{B} = \mu\vec{H}, \phi = BA$

$\mu_0 = 4\pi \times 10^{-7} H/m$

$\phi = \frac{\mathcal{F}}{\Re}$

$\Re = \frac{l}{\mu A}$

$\lambda = N\phi$

$e = \frac{d\lambda}{dt}$

$L = \frac{\lambda}{i} = \frac{N^2}{\Re}$

$W_\phi(\lambda, x) = \frac{1}{2L}\lambda^2$

$W_\phi(i, x) = \frac{1}{2}Li^2$

$f = -\frac{\partial W_\phi(\lambda,x)}{\partial x} = \frac{1}{2}i^2\frac{dL(x)}{dx}$

$f = \frac{\partial W_\phi(i,x)}{\partial x} = \frac{1}{2}i^2\frac{dL(x)}{dx}$

$\lambda_1 = \lambda_{11} + \lambda_{12} = L_{11}i_1 + L_{12}i_2$

$\lambda_2 = \lambda_{21} + \lambda_{22} = L_{21}i_1 + L_{22}i_2$

$e_1 = \frac{d\lambda_1}{dt}$

$e_2 = \frac{d\lambda_2}{dt}$

$W_\phi(\lambda_1, \lambda_2, \theta) = \frac{1}{2}\Gamma_{11}^2\lambda_1^2 + \Gamma_{12}\lambda_1\lambda_2 + \frac{1}{2}\Gamma_{22}\lambda_2^2$

$W_\phi(i_1, i_2, \theta) = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1i_2 + \frac{1}{2}L_{22}i_2^2$

$\tau = -\frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial\theta} = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$

$\tau = \frac{\partial W_\phi(i_1,i_2,\theta)}{\partial\theta} = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$

$e = l\vec{v} \times \vec{B}$

$f = i\vec{l} \times \vec{B}$

$e_a(t) = K_e i_f(t)\omega(t)$

$\tau_d(t) = K_\tau i_f(t) i_a(t)$

$\omega(s) = \frac{K_eI_f(V_a(s)+L_ai_a(0))+(L_as+R_a)(J\omega(0)-\tau_L(s))}{(Js+D)(L_as+R_a)+(K_eI_f)^2}$

$I_a(s) = \frac{(V_a(s)+L_ai_a(0))(Js+D)-K_eI_f(J\omega(0)-\tau_L(s))}{(Js+D)(L_as+R_a)+(K_eI_f)^2}$

$I_f(s) = \frac{V_f(s)+L_fi_f(0)}{L_fs+R_f}$

$Z_Y = \frac{1}{3}Z_\triangle$

$\hat{V}_{an} = \frac{\hat{V}_{ab}}{\sqrt{3}}\angle -30^\circ$

$\hat{I}_{A'} = \sqrt{3}\,\hat{I}_A\angle -30^\circ$

$\triangle$-**Y connection**: $\widehat{E}_{A_1} = a\widehat{E}_{A_2}\angle -30^\circ$, $\widehat{I}'_{A_2} = \frac{1}{a}\widehat{I}_{A_2}\angle -30^\circ$

Y-$\triangle$ connection: $\widehat{E}_{A_1} = a\widehat{E}_{A_2}\angle 30^\circ$, $\widehat{I}'_{A_2} = \frac{1}{a}\widehat{I}_{A_2}\angle 30^\circ$

$\omega_s = \frac{4\pi f}{P}$

$\omega_m = (1-s)\omega_s$

$\widehat{E}'_a = \widehat{E}_a - j\widehat{I}_d(X_d - X_q)$(synchronous generator)

$\widehat{E}'_a = \widehat{E}_a + j\widehat{I}_d(X_d - X_q)$(synchronous motor)

$f_r = sf$

$P_d = 3I_2^2\frac{(1-s)R_2}{s} = \frac{3V_1^2\frac{(1-s)R_2}{s}}{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}$

$s_{\max,p} = \frac{R_2}{R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}$

$P_{d,\max} = \frac{3}{2}\frac{V_1^2}{R_1+R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}$

$\tau_d = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+\frac{R_2}{s}\right)^2+(X_1+X_2)^2\right]}$

$s_{\max,\tau} = \frac{R_2}{\sqrt{R_1^2+(X_1+X_2)^2}}$

$\tau_{d,\max} = \frac{3V_1^2}{2\omega_s\left[R_1+\sqrt{R_1^2+(X_1+X_2)^2}\right]}$

$Z_f = R_f + jX_f = 0.5\frac{jX_m(R_2/s+jX_2)}{R_2/s+j(X_2+X_m)} = 0.5\frac{X_m^2(R_2/s)}{(R_2/s)^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$

$Z_b = R_b + jX_b = 0.5\frac{jX_m(R_2/(2-s)+jX_2)}{R_2/(2-s)+j(X_2+X_m)} = 0.5\frac{X_m^2(R_2/(2-s))}{(R_2/(2-s))^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$

$P_{agf} = I_1^2R_f = 0.5I_{2f}^2\frac{R_2}{s}$

$P_{agb} = I_1^2R_b = 0.5I_{2b}^2\frac{R_2}{2-s}$

$P_{df} = P_{agf} - P_{rcuf} = (1-s)P_{agf}$

$P_{db} = P_{agb} - P_{rcub} = -(1-s)P_{agb}$

$P_d = (1-s)P_{ag}$

$P_{ag} = P_{agf} - P_{agb}$

$P_d = (1-s)P_{ag} = \tau_d\omega_m = (1-s)\tau_d\omega_s$

$\tau_d = \frac{P_{agf}}{\omega_s} - \frac{P_{agb}}{\omega_s} = \tau_{fd} - \tau_{bd}$

$e_a(t) = K_a\Phi_a\omega(t)$

$\tau_d(t) = K_a\Phi_a i_a(t)$

$\theta_m = \frac{2}{P}\theta_e$

$n_m = \frac{1}{NP}n_{pulses}$