Chapter 1. Magnetic Circuit

1.1 Magnetic Circuit

Consider a simple magnetic structure as shown in Figure 1.1. The following assumptions are made for symplifying magnetic circuit analysis:

(A1) The magnetic flux is restricted to flow through the magnetic materials with no leakage;

(A2) The magnetic flux density is uniform within the magnetic materials.

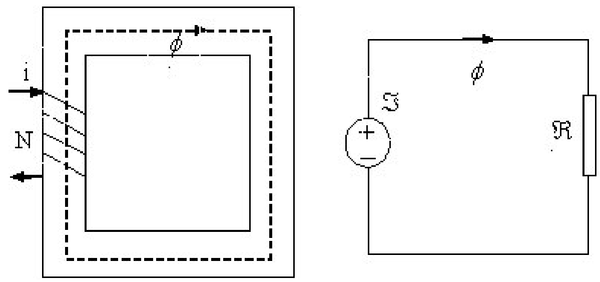


Figure 1.1 Magnetic circuit

If the coil has N turns and carries a current i, the magnetomotive force (mmf) in $A \cdot t$, produced by the current i, is

$$\mathcal{F} = Ni$$

Ampere's law states that the line integral of the tangential component of the magnetic field intensity $\vec{H}$ (in A/m) around a closed path C is equal to the total current passing through the surface enclosed by the path, that is,

$$\mathcal{F} = \oint \vec{H} \cdot d\vec{l} = \oint H \cos\theta dl$$

where $\vec{l}$ is the length vector whose direction is chosen in a way so that the angle between $\vec{H}$ and $d\vec{l}$ is the smallest and θ is the angle between the vectors $\vec{H}$ and $d\vec{l}$. The direction of $\vec{H}$ is detrmined by the right-hand rule:

The right-hand rule 1: Imagine a current-carrying conductor held in the right hand with the thumb pointing in the direction of current flow, the fingers then point in the direction of the magnetic field created by that current.

The right-hand rule 2: If the coil is grasped in the right hand with the fingers pointing in the direction of the current, the thumb will point in the direction of the magnetic field.

Due to (A1), the mean path can be chosen to calculate the magnetic field intensity $\vec{H}$. Note that $\theta = 0$. Thus,

$$\mathcal{F} = Hl$$

where l is the mean legnth of the magnetic core.

The magnetic field intensity $\vec{H}$ is related to the magnetic flux density $\vec{B}$ (in Wb/m^2) by

$$\vec{B} = \mu\vec{H}$$

where $\mu = \mu_r\mu_0$ is called the magnetic permeability (in H/m) with μ_r the ralative permeability and $\mu_0 = 4\pi \times 10^{-7} H/m$ the permeability of the air or free space.

The flux in the core is determined by

$$\phi = BA$$

where A represents the cross-sectional area of the magnetic core.

Therefore, $\mathcal{F} = Hl$ can be rewritten as

$$\mathcal{F} = Hl = \frac{B}{\mu}l = \frac{\frac{\phi}{A}}{\mu}l = \frac{l}{\mu A}\phi = \Re\phi$$

where $\Re = \frac{l}{\mu A}$ is defined as the reluctance of the magnetic circuit.

Comparing the expression $\mathcal{F} = \Re\phi$ with Ohm's law $V = RI$, we find that $\Re$ is analogous to R, ϕ to I, and $\mathcal{F}$ to V. This analogy enables us to represent the magnetic core in terms of an equivalent magnetic circuit as shown in Figure 1.1. Like the voltage source in the electric circuit, the mmf in the magnetic circuit has a polarity. The positive end of the mmf source is the end from which the flux exits and the negative end is the end at which the flux re-enters.

Reluctances in a megnetic circuit obey the same rules as resistances in an electric circuit. The equivalent reluctance of a number of reluctances in series is just the sum of the individual reluctances:

$$\Re_{eq} = \Re_1 + \Re_2 + \cdots$$

Similarly, reluctances in parallel combine according to the equation

$$\frac{1}{\Re_{eq}} = \frac{1}{\Re_1} + \frac{1}{\Re_2} + \cdots$$

Example 1.1:

A magnetic core is shown in Figure 1.2. Its depth is 3cm and its mean length is 30cm. The length of the air-gap is 0.05cm. The coil has 500 turns. The relative permeability of the core is assumed to be 70,000. Neglect fringing effects and assume the flux density of the core is $B_c = 1.0Wb/m^2$. Find the reluctances of the core and air-gap, flux in the core, and the current required.

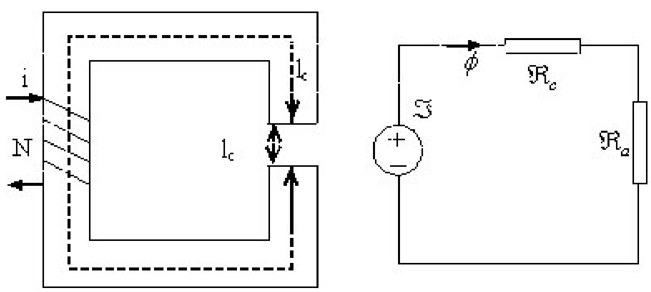


Figure 1.2 Magnetic circuit

Solution: The reluctance of the core is calculated by

$$\Re_c = \frac{l_c}{\mu_r\mu_0 A_c} = \frac{30\times10^{-2}}{70000\times4\pi\times10^{-7}\times0.03\times0.03} = 3789.4A\cdot t/Wb$$

The reluctance of the air-gap is

$$\Re_a = \frac{l_a}{\mu_0 A_a} = \frac{0.05\times10^{-2}}{4\pi\times10^{-7}\times0.03\times0.03} = 442100A\cdot t/Wb$$

The total reluctance in the magnetic circuit is given by

$$\Re = \Re_c + \Re_a = 3789.4 + 442100 = 445890A\cdot t/Wb$$

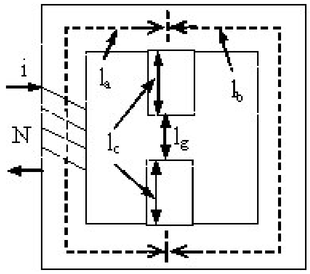
The flux in the magnetic circuit is

$$\phi = B_c A_c = 1.0\times0.03\times0.03 = 0.0009Wb$$

The current in the coil is

$$i = \frac{\phi\Re}{N} = \frac{0.0009\times445890}{500} = 0.8026A$$

Example 1.2: Consider the magnetic circuit as shown in Figure 1.3. Determine the flux through various magnetic paths.



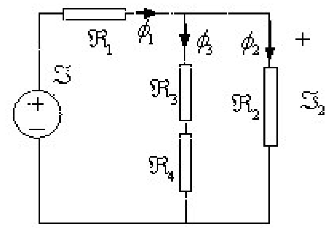


Figure 1.3 Magnetic circuit

Solution: The reluctance in the center leg is $\Re_{center} = \Re_3 + \Re_4$

The total reluctance seen from the coil side is

$$\Re_{total} = \Re_1 + (\Re_3 + \Re_4) \| \Re_2 = \Re 1 + \frac{(\Re_3+\Re_4)\Re_2}{\Re_3+\Re_4+\Re_2} = \frac{\Re_1(\Re_3+\Re_4+\Re_2)+(\Re_3+\Re_4)\Re_2}{\Re_3+\Re_4+\Re_2}$$

The flux in the left leg is

$$\phi_1 = \frac{\mathcal{F}}{\Re_{total}} = \frac{Ni}{\frac{\Re_1(\Re_3+\Re_4+\Re_2)+(\Re_3+\Re_4)\Re_2}{\Re_3+\Re_4+\Re_2}}$$

The flux in the center leg is

$$\phi_3 = \frac{\mathcal{F}_2}{\Re_{center}} = \frac{\phi_1(\Re_{center}\|\Re_2)}{\Re_{center}} = \frac{\phi_1}{\Re_{center}} \frac{\Re_{center}\Re_2}{\Re_{center}+\Re 2} = \frac{\phi_1\Re_2}{\Re_3+\Re_4+\Re_2}$$

The flux in the right leg is

$$\phi_2 = \frac{\mathcal{F}_2}{\Re_2} = \frac{\phi_1(\Re_{center}\|\Re_2)}{\Re_2} = \frac{\phi_1}{\Re_2} \frac{\Re_{center}\Re_2}{\Re_{center}+\Re_2} = \frac{\phi_1(\Re_3+\Re_4)}{\Re_3+\Re_4+\Re_2}$$

where

$$\Re_1 = \frac{l_a}{\mu_r\mu_0 A}, \Re_2 = \frac{l_b}{\mu_r\mu_0 A}, \Re_3 = \frac{2l_c}{\mu_r\mu_0 A}, \Re_4 = \frac{l_g}{\mu_0 A}$$

1.2 AC Excitation, Eddy Current Loss, and Hyteresis Loss

Induced Voltage

Consider the magnetic circuit as shown in Figure 1.1, with the cross-sectional area A and the mean length l. Assume that the flux is a sinusoidal function of time, that is,

$$\phi(t) = \phi_{\max}\sin(\omega t) = AB_{\max}\sin(\omega t)$$

where $\phi_{\max}$ and $B_{\max}$ are the amplitudes of the flux and the flux density, respectively, and $\omega = 2\pi f$.

It follows from Faraday's law that the induced voltage is given by

$$e(t) = N\frac{d\phi}{dt} = N\omega\phi_{\max}\cos(\omega t) = E_{\max}\cos(\omega t)$$

where $E_{\max} = N\omega\phi_{\max} = \omega NAB_{\max} = 2\pi fNAB_{\max}$.

In steady-state operation, we are interested in rms values of voltages and currents. The rms value of the induced voltage is given by

$$E = \frac{2\pi}{\sqrt{2}} fNAB_{\max} = \sqrt{2}\,\pi fNAB_{\max}$$

Excitation Current

To produce a magnetic flux in a magnetic core, a current is required, which is referred to as the excitation current, denoted $i_\phi(t)$. Due to the nonlinearity of the B-H curve and the hysteresis property of the magnetic materials, $i_\phi(t) = \frac{Hl}{N}$ is not a sinusoidal function.

Eddy Current Loss

A time-varying flux induces an emf in the magnetic core in accordance with Faraday's law. Since the magnetic materials are good conductors, the induced emf produces a current along a closed path inside the magnetic core. Such a current is called eddy current because its swirling pattern resembles the eddy current of water.

As a consequence of this eddy current, energy is converted into heat in the resistance of the path, which gives rise to the power loss. Such a loss is referred to as the eddy current loss, which is determined by

$$P_e = k_e f^2 \delta^2 B_{\max}^2 V$$

where P_e is the eddy-current loss in watts (W), k_e is a constant that depends on the conductivity of the magnetic material, f is the frequency in hertz (Hz), δ is the lamination thickness in meters (m), $B_{\max}$ is the maximum flux density in teslas (T), and V is the volume of the magnetic material in cubic meters (m^3).

To reduce the effects of eddy currents, magnetic structures are usually built of thin sheets of laminations of the magnetic material, insulated from each other by an oxide layer or by a thin coat of insulation materials.

Hysteresis Loss

Assume that the flux in the core is initially zero. An ac current is applied to the winding. As the current increases for the first time, the flux in the core traces out path ab as shown in Figure 1.4. However, when the current decreases, the flux traces out a different path bcd, and later when the current increases again, the flux traces out path deb. This failure to retrace flux paths is called hysteresis. The path $bcdeb$ is called a hysteresis loop.

Each time the magnetic materail is made to traverse its hysteresis loop, it produces a power loss, which is commonly referred to as the hysteresis loss. The hysteresis loss can be determined by

$$P_h = k_h f B_{\max}^n V$$

where P_h is the hysteresis loss in watts (W), k_h is a constant that depends on the magnetic material, and n is the Steinmetz exponent.

Core Loss

It is a common practice to lump the eddy current loss and hysteresis loss together to define the core loss

$$P_{core} = P_e + P_h = k_e f^2 \delta^2 B_{\max}^2 V + k_h f B_{\max}^n V = K_e f^2 B_{\max}^2 + K_h f B_{\max}^n$$

where $K_e = k_e \delta^2 V$ and $K_h = k_h V$.

1.3 Flux Linkage, Inductance, and Mutual Inductance

Inductance

Consider the magnetic circuit as shown in Figure 1.1. Faraday's law states that if a flux ϕ passes through a winding of N turns, a voltage will be induced in the winding and the induced voltage e is directly proportional to the rate of change in the flux linkages $\lambda = N\phi$ with respect to time, that is,

$$e = -\frac{d\lambda}{dt} = -N\frac{d\phi}{dt}$$

where the minus sign means that the polarity of the induced voltage is such that if the winding ends were short-circuited, it would produce current that would cause a flux opposing the original flux change.

The self inductance or inductance (in H) of the winding is defined as the ratio of the flux linkages and the current, that is,

$$L = \frac{\lambda}{i} = N\frac{\phi}{i}$$

If L is constant, then

$$e = -\frac{d\lambda}{dt} = -\frac{d(Li)}{dt} = -L\frac{di}{dt}$$

L depends on the physical dimensions of the magnetic circuit and the permeability of the magnetic materials. For the magnetic circuit as shown in Figure 1.1, L can be determined as follows:

$$L = \frac{\lambda}{i} = N\frac{\phi}{i} = N\frac{\frac{\mathcal{F}}{\Re}}{i} = N\frac{N}{\Re} = \frac{N^2}{\Re}$$

Example 1.3: The magnetic circuit of Figure 1.45 consists of an N-turn winding on a magnetic core of infinite permeability with two parallel air gaps of lengths g_1 and g_2 and areas A_1 and A_2, respectively. Find the inductance of the winding and the flux density B_1 in gap 1 when the winding is carrying a current i. Neglect fringing effects at the air gaps.

Solution: The equivalent circuit shows that the total reluctance is equal to the parallel combination of the two gap reluctances $\Re_1 = \frac{g_1}{\mu_0 A_1}$ and $\Re_2 = \frac{g_2}{\mu_0 A_2}$. Thus

$$\lambda = N\phi = N\frac{\mathcal{F}}{\Re} = N\frac{Ni}{\frac{\Re_1\Re_2}{\Re_1+\Re_2}} = \frac{N^2 i(\Re_1+\Re_2)}{\Re_1\Re_2}$$

and

$$L = \frac{\lambda}{i} = \frac{N^2(\Re_1+\Re_2)}{\Re_1\Re_2} = \frac{N^2\left(\frac{g_1}{\mu_0 A_1}+\frac{g_2}{\mu_0 A_2}\right)}{\frac{g_1}{\mu_0 A_1}\frac{g_2}{\mu_0 A_2}} = \mu_0 N^2\left(\frac{A_1}{g_1}+\frac{A_2}{g_2}\right)$$

The flux in gap 1 is

$$\phi_1 = \frac{Ni}{\Re_1} = \frac{\mu_0 A_1 Ni}{g_1}$$

and thus

$$B_1 = \frac{\phi_1}{A_1} = \frac{\mu_0 Ni}{g_1}$$

Mutual Inductance

Consider the magnetic circuit as shown in Figure 1.5. If a current i_1 is applied to coil-1 while a current i_2 to coil-2, then the total mmf is

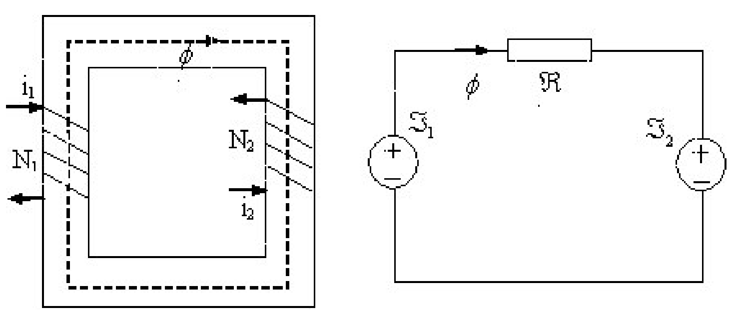


Figure 1.5 Magnetic circuit

$$\mathcal{F} = N_1 i_1 + N_2 i_2$$

The reluctance of the core is

$$\Re = \frac{l}{\mu A}$$

The flux in the core is given by

$$\phi = \frac{\mathcal{F}}{\Re} = \frac{N_1}{\Re}i_1 + \frac{N_2}{\Re}i_2 = N_1\frac{\mu A}{l}i_1 + N_2\frac{\mu A}{l}i_2$$

The flux linkage of coil-1 is

$$\lambda_1 = N_1\phi = N_1^2\frac{\mu A}{l}i_1 + N_1 N_2\frac{\mu A}{l}i_2$$

which can be written

$$\lambda_1 = L_{11}i_1 + L_{12}i_2 = \lambda_{11} + \lambda_{12}$$

where

$$L_{11} = \frac{\lambda_{11}}{i_1} = N_1^2\frac{\mu A}{l}$$

is the self-inductance of coil 1 and $\lambda_{11} = L_{11} i_1$ is the flux linkage of coil-1 due to its own current i_1. The mutual inductance from coil-2 to coil-1 is

$$L_{12} = \frac{\lambda_{12}}{i_2} = N_1 N_2 \frac{\mu A}{l}$$

and $\lambda_{12} = L_{12} i_2$ is the flux linkage of coil-1 due to the current i_2.
Similarly, the flux linkage of coil-2 is

$$\lambda_2 = N_2 \phi = N_1 N_2 \frac{\mu A}{l} i_1 + N_2^2 \frac{\mu A}{l} i_2 = L_{21} i_1 + L_{22} i_2 = \lambda_{21} + \lambda_{22}$$

with $L_{21} = \frac{\lambda_{21}}{i_1} = L_{12} = N_1 N_2 \frac{\mu A}{l}$, $L_{22} = \frac{\lambda_{22}}{i_2} = N_2^2 \frac{\mu A}{l}$, $\lambda_{21} = L_{21} i_2$, and $\lambda_{22} = L_{22} i_2$.
Now suppose $\lambda_{12} = k_1 \lambda_{11}$ and $\lambda_{21} = k_2 \lambda_{22}$. Then it is easily checked that

$$L_{12} L_{21} = \frac{\lambda_{12}}{i_2} \frac{\lambda_{21}}{i_1} = \frac{k_1 \lambda_{11}}{i_2} \frac{k_2 \lambda_{22}}{i_1} = k_1 k_2 L_{11} L_{22}$$

In a linear system, $L_{12} = L_{21} = M$. Therefore,

$$M = k \sqrt{L_{11} L_{22}}$$

where $k = \sqrt{k_1 k_2}$ is known as the coefficient of coupling or the coupling factor between the two coils.
If the inductances are constant, then the induced voltages can be calculated by

$$e_1 = \frac{d\lambda_1}{dt} = \frac{d(L_{11} i_1 + L_{12} i_2)}{dt} = L_{11} \frac{di_1}{dt} + L_{12} \frac{di_2}{dt}$$
$$e_2 = \frac{d\lambda_2}{dt} = \frac{d(L_{21} i_1 + L_{22} i_2)}{dt} = L_{21} \frac{di_1}{dt} + L_{22} \frac{di_2}{dt}$$

Example 1.4: A magnetic circuit with two windings was tested with an ac source at $60Hz$ and the following data were recorded.

Test	Coil Condition	RMS Voltage (V)	RMS Current (A)
1	Coil-1 connected to a voltage source	80	1.5
	Coil-2 open circuit	30	0
2	Coil-2 connected to a voltage source	60	1.0
	Coil-1 open circuit	20	0

Assume the magnetic circuit operated in the linear region and neglect the hysteresis effects. Neglect the winding resistances. Determine the self inductance, mutual inductance, and coupling factor.
Solution: The ac currents can be expressed by $i_j(t) = \sqrt{2} I_j \cos(\omega t)$ with $\omega = 2\pi f$ and $j = 1, 2$.
For the first test, the following equations are obtained:

$$v_1 = L_{11} \frac{di_1}{dt} + L_{12} \frac{di_2}{dt} = L_{11} \frac{di_1}{dt} = -\sqrt{2}\, \omega L_{11} I_1 \sin(\omega t)$$
$$v_2 = L_{21} \frac{di_1}{dt} + L_{22} \frac{di_2}{dt} = L_{21} \frac{di_1}{dt} = -\sqrt{2}\, \omega L_{21} I_1 \sin(\omega t)$$

which implies that the rms values of v_1 and v_2 are equal to $\omega L_{11} I_1$ and $\omega L_{21} I_1$, that is,

$$V_1 = \omega L_{11} I_1 \Rightarrow L_{11} = \frac{V_1}{\omega I_1} = \frac{80}{2\pi \times 60 \times 1.5} = 0.14147H$$
$$V_2 = \omega L_{21} I_1 \Rightarrow L_{21} = \frac{V_2}{\omega I_1} = \frac{30}{2\pi \times 60 \times 1.5} = 53.05mH$$

Similarly, it follows from the second test that

$$v_1 = L_{11} \frac{di_1}{dt} + L_{12} \frac{di_2}{dt} = L_{12} \frac{di_2}{dt} = -\sqrt{2}\, \omega L_{21} I_2 \sin(\omega t)$$
$$v_2 = L_{21} \frac{di_1}{dt} + L_{22} \frac{di_2}{dt} = L_{22} \frac{di_2}{dt} = -\sqrt{2}\, \omega L_{22} I_2 \sin(\omega t)$$

and

$$V_1 = \omega L_{12} I_2 \Rightarrow L_{12} = \frac{V_1}{\omega I_2} = \frac{20}{2\pi \times 60 \times 1.0} = 53.05mH$$
$$V_2 = \omega L_{22} I_2 \Rightarrow L_{22} = \frac{V_2}{\omega I_2} = \frac{60}{2\pi \times 60 \times 1.0} = 0.15915H$$

The coupling factor is

$k = \frac{L_{12}}{\sqrt{L_{11}L_{22}}} = \frac{53.05\times10^{-3}}{\sqrt{0.14147\times0.15915}} = 0.35355$

Example 1.5: Given the magnetic circuit as shown in Figure 1.6, neglect fringing effects, leakage flux and reluctances in the magnetic materials. Determine the self-inductances and mutual inductances.

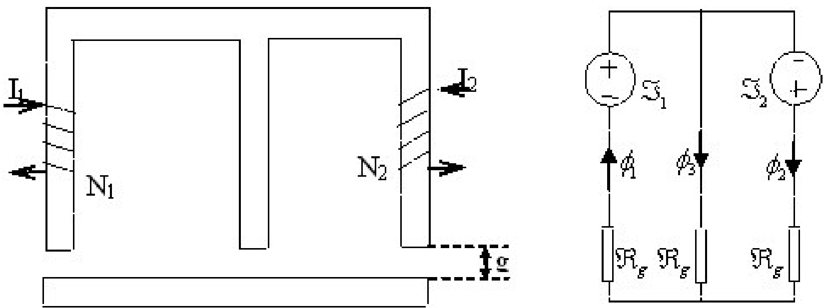


Figure1.6 The magnetic circuit for Example 1.5

Solution: The reluctance of each air gap is $\Re_g = \frac{g}{\mu_0 A}$ where A is the cross-sectional area of the gap. The fluxes satisfy the following equation

$$\phi_1 = \phi_2 + \phi_3$$

For the left loop, we have

$$\mathcal{F}_1 = \phi_1\Re_g + \phi_3\Re_g = \Re_g(\phi_1 + \phi_3)$$

that is,

$$\phi_1 + \phi_3 = \frac{\mathcal{F}_1}{\Re_g}$$

For the right loop, we have

$$\mathcal{F}_2 = \phi_2\Re_g - \phi_3\Re_g = \Re_g(\phi_2 - \phi_3)$$

that is,

$$\phi_2 - \phi_3 = \frac{\mathcal{F}_2}{\Re_g}$$

Substituting $\phi_1 = \phi_2 + \phi_3$ into $\phi_1 + \phi_3 = \frac{\mathcal{F}_1}{\Re_g}$ gives

$$\phi_2 + 2\phi_3 = \frac{\mathcal{F}_1}{\Re_g}$$

Subtracting $\phi_2 - \phi_3 = \frac{\mathcal{F}_2}{\Re_g}$ from $\phi_2 + 2\phi_3 = \frac{\mathcal{F}_1}{\Re_g}$ yields

$$3\phi_3 = \frac{\mathcal{F}_1}{\Re_g} - \frac{\mathcal{F}_2}{\Re_g} = \frac{\mathcal{F}_1 - \mathcal{F}_2}{\Re_g}$$

that is,

$$\phi_3 = \frac{\mathcal{F}_1 - \mathcal{F}_2}{3\Re_g}$$

Then, it follows from $\phi_2 - \phi_3 = \frac{\mathcal{F}_2}{\Re_g}$ that

$$\phi_2 = \phi_3 + \frac{\mathcal{F}_2}{\Re_g} = \frac{\mathcal{F}_1 - \mathcal{F}_2}{3\Re_g} + \frac{3\mathcal{F}_2}{3\Re_g} = \frac{\mathcal{F}_1 + 2\mathcal{F}_2}{3\Re_g} = \frac{N_1 i_1 + 2N_2 i_2}{3\Re_g}$$

and from $\phi_1 = \phi_2 + \phi_3$, we have

$$\phi_1 = \phi_2 + \phi_3 = \frac{\mathcal{F}_1 + 2\mathcal{F}_2}{3\Re_g} + \frac{\mathcal{F}_1 - \mathcal{F}_2}{3\Re_g} = \frac{2\mathcal{F}_1 + \mathcal{F}_2}{3\Re_g} = \frac{2N_1 i_1 + N_2 i_2}{3\Re_g}$$

Therefore,

$$\lambda_1 = N_1\phi_1 = \frac{2N_1^2}{3\Re_g} i_1 + \frac{N_1 N_2}{3\Re_g} i_2$$

$$\lambda_2 = N_2\phi_2 = \frac{N_1 N_2}{3\Re_g} i_1 + \frac{2N_2^2}{3\Re_g} i_2$$

which implies that

$L_{11} = \frac{2N_1^2}{3\Re_g}, L_{12} = L_{21} = \frac{N_1 N_2}{3\Re_g}, L_{22} = \frac{2N_2^2}{3\Re_g}$

with $\Re_g = \frac{g}{\mu_0 A}$.

Chapter 2. Electromechanical Energy Conversion

An electromechanical system consists of an electrical subsystem (electric circuits such as windings), a magnetic subsystem (magnetic field in the magnetic cores and airgaps), and a mechanical subsystem (mechanically movable parts such as a plunger in a linear actuator and a rotor in a rotating electrical machine), as shwon in Figure 2.1.

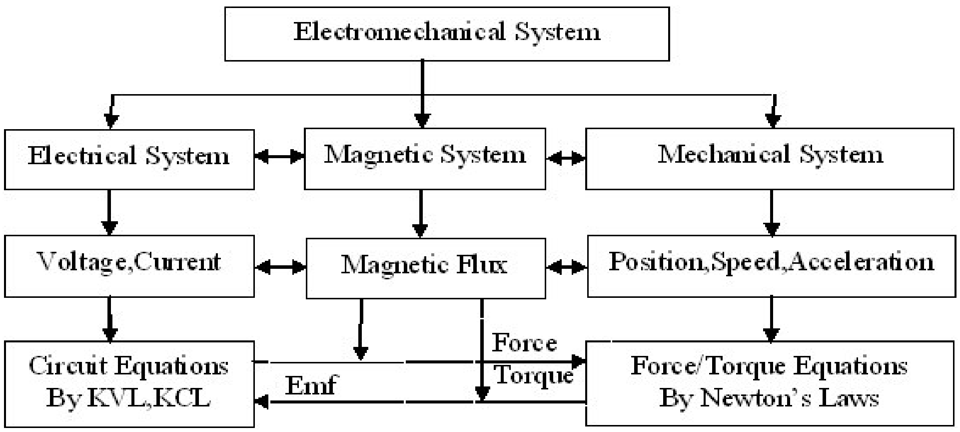


Figure 2.1 General concept of electromechanical system modeling

2.1 Force and Torque on a Current Carrying Conductor: Motor Action

The force on a conductor carrying a current i in a uniform magnetic field $\vec{B}$ is given by the Lorentz's force law:

$$\vec{f} = i\vec{l} \times \vec{B}$$

In a rotating system, the torque about an axis can be calculated by

$$\tau = \vec{r} \times \vec{f}$$

where $\vec{r}$ is the radius vector from the axis towards the conductor.

2.2 Energy Stored in Magnetic Field

Energy Stored in Magnetic Circuit with a Single Coil

Consider the magnetic circuit with a single winding as shown in Figure 1.1. Neglect losses. Note that

$$e = \frac{d\lambda}{dt}$$

and

$$L = \frac{\lambda}{i}$$

The electric input power is determined from

$$p = ie = i\frac{d\lambda}{dt}$$

The energy stored in the field during dt is

$$dW_\phi = pdt = id\lambda$$

With zero initial energy stored in the magnaetic field, the energy at time t is

$$W_\phi = \int_0^t pdt = \int_0^\lambda id\lambda = \int_0^\lambda \frac{\lambda}{L} d\lambda = \frac{1}{2L}\lambda^2$$

or

$$W_\phi = \int_0^t pdt = \int_0^\lambda id\lambda = \int_0^i id(Li) = \frac{1}{2}Li^2$$

Example 2.1: (see Example 1.1) A magnetic core is shown in Figure 1.3. Its depth is 3cm and its mean length is 30cm. The length of the air-gap is 0.05cm. The coil has 500 turns. The relative permeability of the core is assumed to be 70,000. Neglect fringing effects and assume the flux density of the core is $B_c = 1.0Wb/m^2$. The frequency of the source is $60Hz$. Find the inductances of the core and energy stored in the field.

Solution: It follows from Example 1.1 that

$$\Re_c = \frac{l_c}{\mu_r \mu_0 A_c} = \frac{30\times10^{-2}}{70000\times4\pi\times10^{-7}\times0.03\times0.03} = 3789.4A \cdot t/Wb$$

$$\Re_a = \frac{l_a}{\mu_0 A_a} = \frac{0.05\times10^{-2}}{4\pi\times10^{-7}\times0.03\times0.03} = 442100A \cdot t/Wb$$

$$\Re = \Re_c + \Re_a = 3789.4 + 442100 = 445890A \cdot t/Wb$$

The inductance is

$$L = \frac{\lambda}{i} = \frac{N\phi}{i} = \frac{N}{i}\frac{\mathcal{F}}{\Re} = \frac{N}{i}\frac{Ni}{\Re} = \frac{N^2}{\Re} = \frac{500^2}{445890} = 0.56068H$$

If $\Re_c$ is neglected,

$$L = \frac{N^2}{\Re_a} = \frac{500^2}{442100} = 0.56548H$$

The error caused by neglecting the reluctance of the core is only

$$error = 0.56548 - 0.56068 = 0.0048H.$$

The flux in the magnetic circuit is

$$\phi = B_c A_c = 1.0 \times 0.03 \times 0.03 = 0.0009Wb$$

The current in the coil is

$$i = \frac{\phi\Re}{N} = \frac{0.0009\times445890}{500} = 0.8026A$$

The stored energy is

$$W_\phi = \frac{1}{2}Li^2 = \frac{1}{2} \times 0.56068 \times (0.8026)^2 = 0.18059J$$

Energy Stored in Magnetic Circuit with Two Coils

Consider the magnetic circuit with two windings as shown in Figure 1.5. Neglect losses. Then the electric input energy is equal to the energy stored in the field, that is,

$$dW_s = dW_\phi$$

The electric input power is

$$p = e_1 i_1 + e_2 i_2$$

and the input energy is

$$dW_e = pdt = e_1 i_1 dt + e_2 i_2 dt$$

Note that $e_1 = \frac{d\lambda_1}{dt}$ and $e_2 = \frac{d\lambda_2}{dt}$.

Thus, the stored energy can be expressed as

$$dW_\phi = dW_e = i_1 d\lambda_1 + i_2 d\lambda_2$$

Recall the relations $L_{12} = L_{21}$ and

$$\lambda_1 = \lambda_{11} + \lambda_{12} = L_{11} i_1 + L_{12} i_2$$

$$\lambda_2 = \lambda_{21} + \lambda_{22} = L_{21} i_1 + L_{22} i_2$$

to get

$$\begin{aligned} dW_\phi &= dW_e = i_1 d(L_{11}i_1 + L_{12}i_2) + i_2 d(L_{21}i_1 + L_{22}i_2) \\ &= L_{11}i_1 di_1 + L_{12}i_1 di_2 + L_{21}i_2 di_1 + L_{22}i_2 di_2 \\ &= L_{11}i_1 di_1 + L_{12}(i_1 di_2 + i_2 di_1) + L_{22}i_2 di_2 \\ &= L_{11}i_1 di_1 + L_{12}d(i_1 i_2) + L_{22}i_2 di_2 \end{aligned}$$

For the case that the inductances are independdent of currents, W_ϕ can be calculated by

$$W_\phi = \int (L_{11}i_1 di_1 + L_{12}d(i_1 i_2) + L_{22}i_2 di_2) = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1 i_2 + \frac{1}{2}L_{22}i_2^2$$

On the other hand, solving the equations

$$\lambda_1 = L_{11}i_1 + L_{12}i_2$$
$$\lambda_2 = L_{21}i_1 + L_{22}i_2$$

for i_1 and i_2 yields

$$i_1 = \Gamma_{11}\lambda_1 + \Gamma_{12}\lambda_2$$
$$i_2 = \Gamma_{21}\lambda_1 + \Gamma_{22}\lambda_2$$

where $\Gamma_{11} = L_{22}/\Delta, \Gamma_{12} = \Gamma_{21} = -L_{12}/\Delta, \Gamma_{22} = L_{11}/\Delta,$ and $\Delta = L_{11}L_{22} - (L_{12})^2$.
Then,

$$\begin{aligned} dW_\phi &= (\Gamma_{11}\lambda_1 + \Gamma_{12}\lambda_2)d\lambda_1 + (\Gamma_{21}\lambda_1 + \Gamma_{22}\lambda_2)d\lambda_2 \\ &= \Gamma_{11}\lambda_1 d\lambda_1 + \Gamma_{12}\lambda_2 d\lambda_1 + \Gamma_{21}\lambda_1 d\lambda_2 + \Gamma_{22}\lambda_2 d\lambda_2 \\ &= \Gamma_{11}\lambda_1 d\lambda_1 + \Gamma_{12}(\lambda_2 d\lambda_1 + \lambda_1 d\lambda_2) + \Gamma_{22}\lambda_2 d\lambda_2 \\ &= \Gamma_{11}\lambda_1 d\lambda_1 + \Gamma_{12}d(\lambda_1\lambda_2) + \Gamma_{22}\lambda_2 d\lambda_2 \end{aligned}$$

which means that

$$W_\phi = \int (\Gamma_{11}\lambda_1 d\lambda_1 + \Gamma_{12}d(\lambda_2\lambda_1) + \Gamma_{22}\lambda_2 d\lambda_2) = \frac{1}{2}\Gamma_{11}^2\lambda_1^2 + \Gamma_{12}\lambda_1\lambda_2 + \frac{1}{2}\Gamma_{22}\lambda_2^2$$

2.3 Force and Torque Calculation from Energy

A Singly Excited Linear Actuator

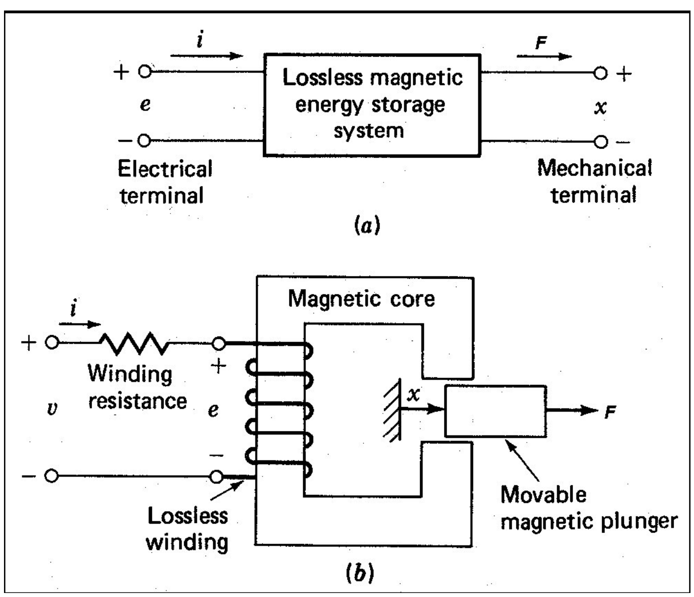


Figure2.2 A singly excited linear actuator

Consider a singly excited linear actuator as shown in Figure 2.3. The winding resistance is

R. A voltage v is applied to the winding, which produces a current i. Assume that at a certain time instant t, the movable plunger is positioned at x and the force acting on the plunger is $\vec{f}$ with the reference direction chosen in the positive direction of the x axis, as shown in the diagram. After a time interval dt, the plunger has moved for a distance dx under the action of the force $\vec{f}$. The mechanical work done by the force during this time interval is thus

$$dW_m = fdx$$

The electrical energy supplied by the electrical source during this time interval is calculated by

$$dW_e = vidt$$

The energy dissipated in the winding resistance during this time interval is

$$dW_{loss} = Ri^2dt$$

Suppose that there is no mechanical losses in the system. According to the principle of conservation of energy (energy is neither created nor destroyed and it is merely changed in form), the energy stored in the magnetic field during this time interval dW_f must satisfy

$$dW_\phi = dW_e - dW_{loss} - dW_m = vidt - Ri^2dt - fdx = (vi - ri)idt - fdx = eidt - fdx = \frac{d\lambda}{dt}idt - fdx = id\lambda$$

From the above equation, we know that the energy stored in the magnetic field W_f is a function of λ and x. Therefore, W_f can be expressed as

$$W_\phi(\lambda, x) = \frac{\partial W_\phi(\lambda,x)}{\partial \lambda}d\lambda + \frac{\partial W_\phi(\lambda,x)}{\partial x}dx$$

By comparing the above two equations, we conclude

$$i = \frac{\partial W_\phi(\lambda,x)}{\partial \lambda}, f = -\frac{\partial W_\phi(\lambda,x)}{\partial x}$$

It follows from Section 2.3 that the energy stored in the magnetic field can be calculated by

$$W_\phi(\lambda, x) = \int_0^\lambda i(\lambda, x)d\lambda$$

For a magnetically linear system (with a constant permeability or a straight line magnetization curve such that the inductance of the coil is independent of the excitation current), the above expression becomes

$$W_\phi(\lambda, x) = \frac{1}{2}\frac{\lambda^2}{L(x)}$$

Therefore, the force can be calculated by

$$f = -\frac{\partial W_\phi(\lambda,x)}{\partial x} = \frac{1}{2}\left(\frac{\lambda}{L(x)}\right)^2\frac{dL(x)}{dx} = \frac{1}{2}i^2\frac{dL(x)}{dx}$$

Example 2.2: Calculate the force acting on the plunger of a linear actuator as shown in Figure 2.3.

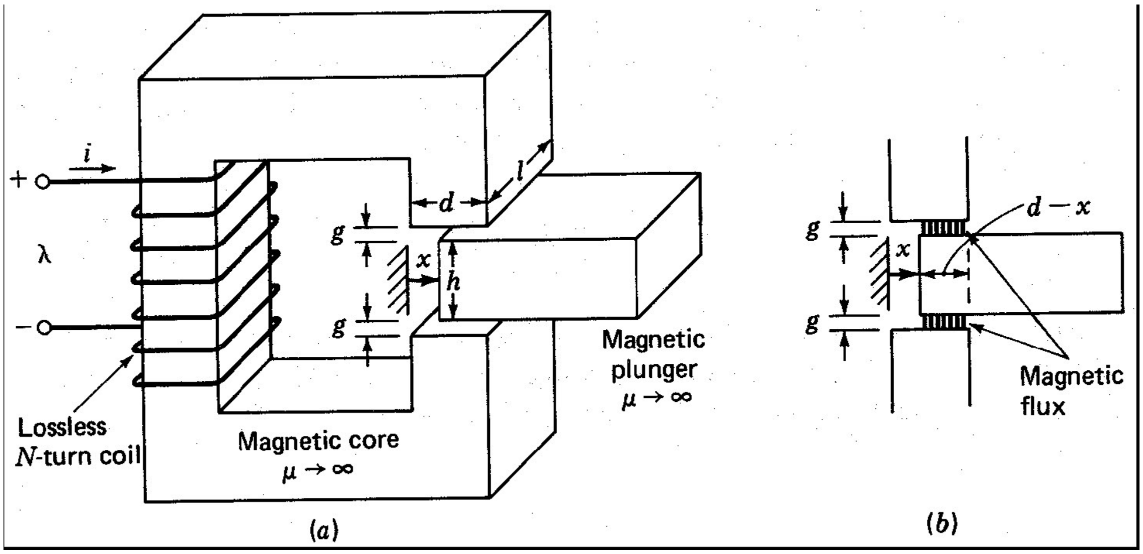


Figure2.3 A singly excited linear actuator

Solution: The reluctance of the actuator is

$$\Re_g = \frac{2g}{\mu_0 (d-x) l}$$

The inductance of the actuator is

$$L(x) = \frac{N^2}{\Re_g} = \frac{\mu_0 N^2 l}{2g}(d-x)$$

Therefore, the force acting on the plunger is

$$f = \frac{1}{2} i^2 \frac{dL(x)}{dx} = -\frac{\mu_0 l}{4g}(Ni)^2$$

The minus sign of the force indicates that the direction of the force is to reduce the displacement so as to reduce the reluctance of the air gaps. Since this force is caused by the variation of magnetic reluctance of the magnetic circuit, it is known as the reluctance force.

Doubly Excited Rotating Actuator

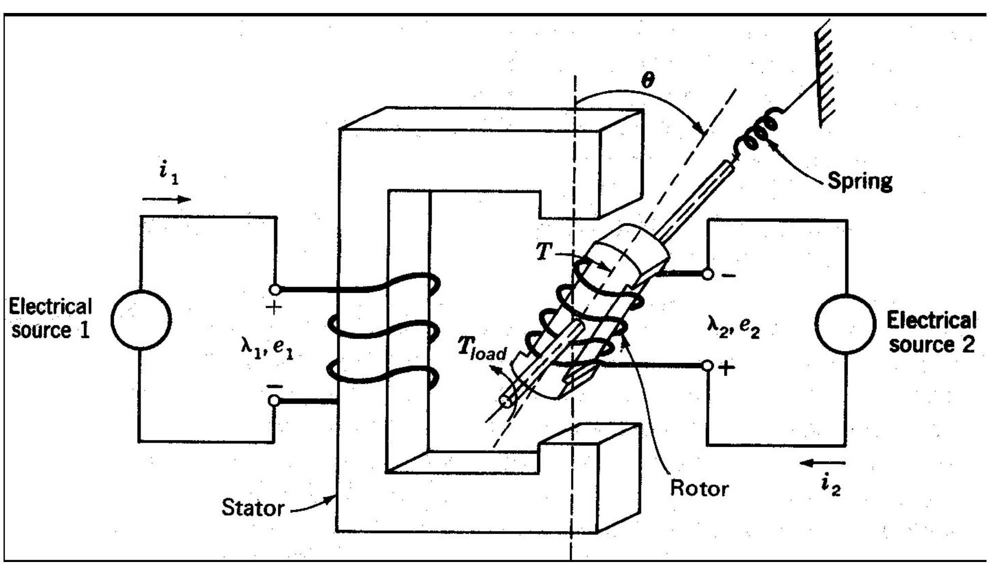


Figure 2.4 A doubly excited actuator

The general principle for force and torque calculation discussed above is equally applicable to multi-excited systems. Consider a doubly excited rotating actuator as shown in Figure 2.4. The differential energy functions can be derived as following:

$$dW_\phi = dW_e - dW_m$$

where

$$dW_e = i_1 d\lambda_1 + i_2 d\lambda_2$$

$$dW_m = \tau d\theta$$

Hence,

$$dW_\phi(\lambda_1,\lambda_2,\theta) = i_1 d\lambda_1 + i_2 d\lambda_2 - \tau d\theta = \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \lambda_1} d\lambda_1 + \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \lambda_2} d\lambda_2 + \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \theta} d\theta$$

which implies that

$$i_1 = \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \lambda_1}$$

$$i_2 = \frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \lambda_2}$$

$$\tau = -\frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \theta}$$

Note that for a linear system

$$W_\phi(\lambda_1,\lambda_2,\theta) = \frac{1}{2}\Gamma_{11}^2\lambda_1^2 + \Gamma_{12}\lambda_1\lambda_2 + \frac{1}{2}\Gamma_{22}\lambda_2^2$$

Then, we have

$$\tau = -\frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial \theta} = -\left(\frac{1}{2}\lambda_1^2\frac{d\Gamma_{11}(\theta)}{d\theta} + \lambda_1\lambda_2\frac{d\Gamma_{12}(\theta)}{d\theta} + \frac{1}{2}\lambda_2^2\frac{d\Gamma_{22}(\theta)}{d\theta}\right)$$

It is useful to express τ in terms of L_{11}, L_{12}, and L_{22}. It can be verified that

$$\tau = \frac{\partial W_\phi(i_1,i_2,\theta)}{\partial \theta} = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1 i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$$

Example 2.3: Write an expression for the inductance of the magnetic circuit for Figure 2.5 as a function of θ and derive an expression for the torque acting on the robot as a function of the winding current i and the rotor angle θ. Neglect the effects of fringing and the reluctance of the steel.

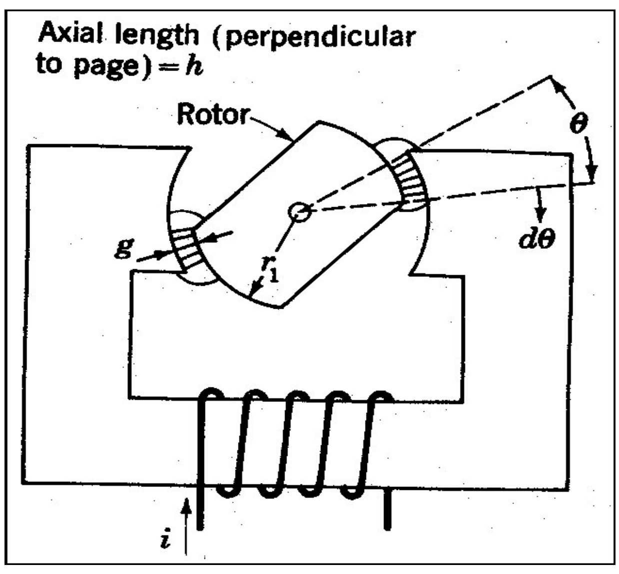


Figure2.5 Singly excited rotational actuator

Solution: The reluctance of the air-gaps is

$$\Re_g = \frac{2g}{\mu_0 h(r+0.5g)\theta}$$

The inductance of the magnetic circuit is

$$L(\theta) = \frac{N^2}{\Re_g} = \frac{\mu_0 N^2 h(r+0.5g)}{2g}\theta$$

The energy stored in the magnetic field is

$$W_\phi(\lambda,\theta) = \int_0^\lambda i(\lambda,\theta)d\lambda = \int_0^\lambda \frac{\lambda}{L(\theta)}d\lambda = \frac{1}{2}\frac{\lambda^2}{L(\theta)}$$

The torque is

$$\tau = -\frac{\partial W_\phi(\lambda,\theta)}{\partial\theta} = \frac{1}{2}\left(\frac{\lambda}{L(\theta)}\right)^2\frac{dL(\theta)}{d\theta} = \frac{1}{2}i^2\frac{dL(\theta)}{d\theta} = \frac{1}{2}\frac{\mu_0 N^2 h(r+0.5g)}{2g}i^2$$

Example 2.4: In the system shown in Figure 2.4, the inductances in henrys are given as

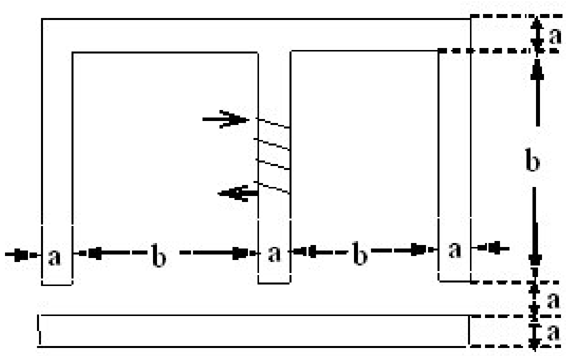
$$L_{11} = 0.001(3+\cos 2\theta)$$
$$L_{12} = 0.3\cos\theta$$
$$L_{22} = 30+10\cos 2\theta$$

Find the torque $\tau(\theta)$for currents $i_1 = 0.8A$ and $i_2 = 0.01A$.

Solution: The torque can be determined by

$$\tau = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1 i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$$
$$= \frac{1}{2}i_1^2(-0.002\sin 2\theta) + i_1 i_2(-0.3\sin\theta) + \frac{1}{2}i_2^2(-20\sin 2\theta)$$
$$= -0.001 i_1^2\sin 2\theta - 0.3 i_1 i_2\sin\theta - 10 i_2^2\sin 2\theta$$
$$= -0.001\times 0.8^2\sin 2\theta + 0.3\times 0.8\times 0.01\sin\theta - 10\times 0.01^2\sin 2\theta$$
$$= -0.00164\sin 2\theta - 0.0024\sin\theta$$

Example 2.5: The magnetic circuit of Figure 2.6 is excited by a 100-turn coil wound over the central leg. The depth is 1cm, a=1cm and b=5cm. Determine the current in the coil that is necesssary to keep the movable part suspended at a distance of 1cm. Both magnetic circuit and movable part have a cross-sectional area of 1cm^2. What is the energy stored in the systems? The relative permeability and the density of the magnetic material are 2000 and 7.85g/cm^3, respectively.



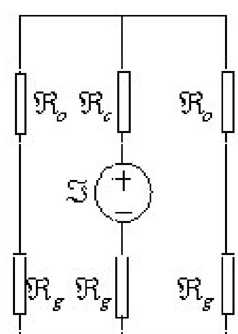


Figure2.6 Figure for Example 2.5

Solution: The mean length for each of the outer legs including a part of the movable part is

$$l_o = \frac{1}{2}a+b+\frac{1}{2}a+\frac{1}{2}a+b+\frac{1}{2}a+\frac{1}{2}a+b+\frac{1}{2}a = 3a+3b = 3(a+b) = 3(1+5) = 18cm$$

The mean length of the central leg is

$$l_c = \frac{1}{2}a+b+\frac{1}{2}a = a+b = 1+5 = 6cm$$

The length of the air gap is assumed to be x. The reluctance of each par is calculated as

$$\Re_o = \frac{l_o}{\mu_r\mu_0 A} = \frac{18\times10^{-2}}{2000\times4\pi\times10^{-7}\times0.0001} = 7.1620\times10^5 A\cdot t/Wb$$
$$\Re_c = \frac{l_c}{\mu_r\mu_0 A} = \frac{6\times10^{-2}}{2000\times4\pi\times10^{-7}\times0.0001} = 2.3873\times10^5 A\cdot t/Wb$$
$$\Re_g = \frac{l_c}{\mu_r\mu_0 A} = \frac{x}{4\pi\times10^{-7}\times0.0001} = 7.9577\times10^9 x$$

The applied mmf is $\mathcal{F} = Ni = 100i$ where i is the required current in the coil.
The total reluctance as viewed from the magnetomotive source is

$$\begin{aligned}\Re &= \Re_c + \Re_g + 0.5(\Re_o + \Re_g) \\ &= 2.3873 \times 10^5 + 7.9577 \times 10^9 x + 0.5(7.1620 \times 10^5 + 7.9577 \times 10^9 x) \\ &= 1.1937 \times 10^{10} x + 5.9683 \times 10^5\end{aligned}$$

Hence, the inductance is

$$L(x) = \frac{N^2}{\Re} = \frac{100^2}{1.1937\times10^{10}x+5.9683\times10^5} = \frac{1}{1.1937\times10^6x+59.683}$$

The megnetic force acting on the movable part is

$$f = \frac{1}{2}i^2\frac{dL(x)}{dx} = -\frac{1}{2}i^2\frac{1.1937\times10^6}{\left(59.683+1.1937\times10^6x\right)^2}$$

The negative sign indicates that the force is acting in the upward direction. Therefore, the magnitude of the force of attraction for $x = 1cm$ is

$$f = -\frac{1}{2}\frac{1.1937\times10^6}{\left(59.683+1.1937\times10^6\times0.01\right)^2}i^2 = 4.1471 \times 10^{-3}i^2$$

The length of the movable part is $3a + 2b = 13cm$. The volume of the movable part is $13 \times 1 = 13cm^3$, so the mass of the movable part is $13 \times 7.85 = 102.05g$.
For the movable part to be stationary, the force of gravity must equal to the magnetic force calculated by

$$f_g = mg = 102.05 \times 10^{-3} \times 9.8 = 1.0001N$$

that is

$$4.1471 \times 10^{-3}i^2 = 1.0001$$

Solving this equation for the current gives

$$i = \sqrt{\frac{1.0001}{4.1471\times10^{-3}}} = 15.529A$$

The inductance of the magnetic circuit at $x = 1cm$ is

$$L(1cm) = \frac{1}{1.1937\times10^6\times0.01+59.683} = 8.3356 \times 10^{-5}H$$

The energy stored in the magnetic field is

$$W_f = \frac{1}{2}Li^2 = \frac{1}{2} \times 8.3356 \times 10^{-5} \times 15.529^2 = 1.0051 \times 10^{-2}J$$

Chapter 3 Dynamics of Electromechanical Systems

3.1 Mathematical Model

Figure 3.1 shows the model of a simple electromechanical system, which consists of three parts: an electrical system, an electromechanical energy-conversion system, and an mechanical system.

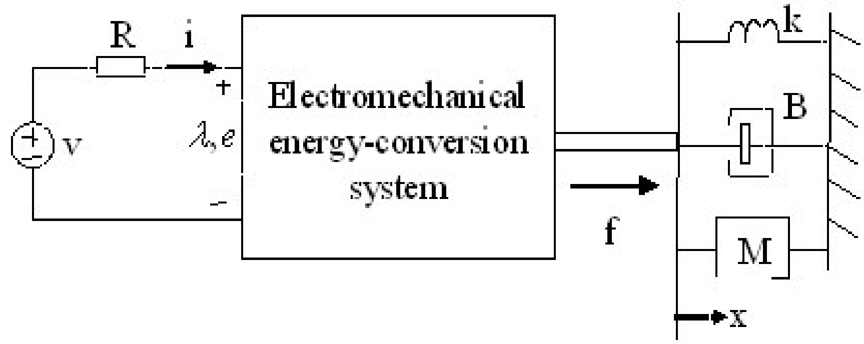


Figure3.1 Model of an electromechanical system

Neglect losses in the electromechanical system. For the electrical system, the following equation can be obtained from KVL:

$$v = Ri + e = Ri + \frac{d\lambda}{dt} = Ri + \frac{d(L(x)i)}{dt} = Ri + L(x)\frac{di}{dt} + i\frac{dL(x)}{dx}\frac{dx}{dt}$$

Assume that the spring is normally unstretched at $x = 0$. Then, the following equation can be obtained from Newton's law:

$$f - kx - B\frac{dx}{dt} = M\frac{d^2x}{dt^2}$$

where f and $L(x)$ depend on the properties of the electromechanical energy-conversion system.

The diffential equations above are called the mathematical model of the electromechanical system.

Example 3.1: An electromechanical system is shown in Figure 3.2. The voltage source has a DC voltage V_s. The switch is turned on at $t = 0$. The bar slides along a pair of frictionless rails in a horizontal plane. The bar has a mass of m. The resistance of the system is R. Assume all initial conditions are zero. Determine the current $i(t)$ and the velocity $v = \frac{dx}{dt}$ of the bar.

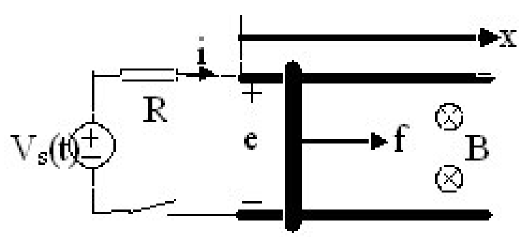


Figure3.2 Example 3.1

Solution: The induced voltage is

$$e(t) = l\vec{v} \times \vec{B} = lBv(t)$$

From KVL, we obtain

$$v_s(t) = Ri(t) + e(t) = Ri(t) + lBv(t)$$

which implies that

$$i(t) = \frac{1}{R}v_s(t) - \frac{lB}{R}v(t)$$

The induced force is

$$f = i\,\vec{l} \times \vec{B} = lBi(t) = \frac{lB}{R}v_s(t) - \frac{l^2B^2}{R}v(t)$$

From Newton's law, we have

$$f = \frac{lB}{R}v_s(t) - \frac{l^2B^2}{R}v(t) = m\frac{dv(t)}{dt}$$

So the mathematical model for this system is

$$m\frac{dv(t)}{dt} + \frac{l^2B^2}{R}v(t) = \frac{lB}{R}v_s(t)$$

This equation can be solved by using Laplace transform. Note that $v_s(t)$ is a step signal and its Laplace transform is $\frac{V_s}{s}$. Taking Laplace transform gives

$$msV(s) + \frac{l^2B^2}{R}V(s) = \frac{lB}{R}\frac{V_s}{s}$$

Solving it for $V(s)$ yields

$$V(s) = \frac{\frac{lB}{R}\frac{V_s}{s}}{ms+\frac{l^2B^2}{R}} = \frac{\frac{lBV_s}{mR}}{s\left(s+\frac{l^2B^2}{mR}\right)}$$

Carrying the partial fraction expansion gives

$$V(s) = \frac{A_1}{s} + \frac{A_2}{s+\frac{l^2B^2}{mR}}$$

where

$$A_1 = s\frac{\frac{lBV_s}{mR}}{s\left(s+\frac{l^2B^2}{mR}\right)}\Bigg|_{s=0} = \frac{\frac{lBV_s}{mR}}{\frac{l^2B^2}{mR}} = \frac{V_s}{lB}$$

$$A_2 = \left(s + \frac{l^2B^2}{mR}\right)\frac{\frac{lBV_s}{mR}}{s\left(s+\frac{l^2B^2}{mR}\right)}\Bigg|_{s=-\frac{l^2B^2}{mR}} = \frac{\frac{lBV_s}{mR}}{-\frac{l^2B^2}{mR}} = -\frac{V_s}{lB}$$

Therefore,

$$V(s) = \frac{\frac{V_s}{lB}}{s} + \frac{-\frac{V_s}{lB}}{s+\frac{l^2B^2}{mR}}$$

Taking inverse Laplace transform gives

$$v(t) = \frac{V_s}{lB} - \frac{V_s}{lB}e^{-\frac{l^2B^2}{mR}t}$$

The current in the circuit is given by

$$i(t) = \frac{1}{R}v_s(t) - \frac{lB}{R}v(t) = \frac{V_s}{R} - \frac{lB}{R}\left(\frac{V_s}{lB} - \frac{V_s}{lB}e^{-\frac{l^2B^2}{mR}t}\right) = \frac{V_s}{R}e^{-\frac{l^2B^2}{mR}t}$$

3.2 Dynamics of DC Generators

A separetely excited dc generator delivering power to a static load is shown in Figure 3.3. Assume that the speed of the generator is constant.

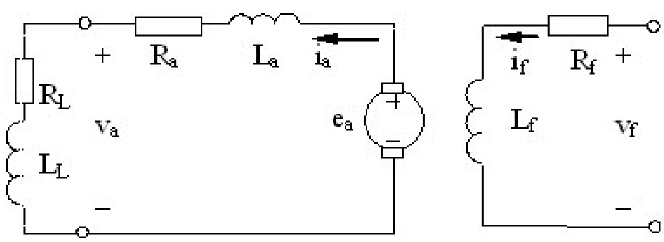


Figure3.3 Equivalent circuit of a dc generator

During the transient state, the field voltage satisfies the equation

$$V_f = R_f i_f(t) + L_f\frac{di_f(t)}{dt}$$

and the generated voltage is

$$e_a(t) = K_e\omega i_f(t) = (R_a + R_L)i_a(t) + (L_a + L_L)\frac{di_a(t)}{dt}$$

Taking the Laplace transform gives

$$V_f(s) = R_f I_f(s) + L_f[sI_f(s) - i_f(0)]$$
$$K_e\omega I_f(s) = (R_a + R_L)I_a(s) + (L_a + L_L)[sI_a(s) - i_a(0)]$$

Solving these equations yields

$$I_f(s) = \frac{V_f(s)+L_f i_f(0)}{L_f s+R_f}$$
$$I_a(s) = \frac{K_e\omega I_f(s)+(L_a+L_L)i_a(0)}{(L_a+L_L)s+R_a+R_L} = \frac{K_e\omega(V_f(s)+L_f i_f(0))+(L_a+L_L)i_a(0)(L_f s+R_f)}{(L_f s+R_f)((L_a+L_L)s+R_a+R_L)}$$

Note that when the system reaches its steady state condition, $\frac{di_f(t)}{dt} = 0$ and $\frac{di_a(t)}{dt} = 0$, from which the following equations are obtained for steady state operation:

$$V_f = R_f i_f(\infty)$$
$$K_e\omega i_f(\infty) = (R_a + R_L)i_a(\infty)$$

that is,

$$i_f(\infty) = \frac{V_f}{R_f}$$
$$i_a(\infty) = \frac{K_e\omega i_f(\infty)}{R_a+R_L}$$

Example 3.2: A separately excited dc generator operating at 1500rpm has the following parameters: $R_a = 0.2\Omega$, $L_a = 2.5mH$, $R_f = 3\Omega$, $L_f = 25mH$, and $K_e = 0.191$. If a dc voltage of 120V is suddenly applied to the field winding under a load with $R_L = 40\Omega$ and $L_L = 40mH$, determine the field current, armature current, and generated voltage as a function of time, the approximate time to reach the steady-state condition, and the steady-state values of the field current and induced voltage.

Solution: The Laplace transform of the field voltage is $V_f(s) = \frac{120}{s}$. The initial conditions are $i_f(0) = 0$ and $i_a(0) = 0$. The field current in s-domain is given by

$$I_f(s) = \frac{V_f(s)+L_f i_f(0)}{L_f s+R_f} = \frac{\frac{120}{s}+0.025\times 0}{0.025s+3} = \frac{120}{s(0.025s+3)} = \frac{\frac{120}{0.025}}{s\left(s+\frac{3}{0.025}\right)} = \frac{4800}{s(s+120)} = \frac{A}{s} + \frac{B}{s+120} = \frac{40}{s} + \frac{-40}{s+120}$$

where

$$A = s\frac{4800}{s(s+120)}\bigg|_{s=0} = \frac{4800}{0+120} = 40$$
$$B = (s+120)\frac{4800}{s(s+120)}\bigg|_{s=-120} = \frac{4800}{-120} = -40$$

Therefore, the field current in time domain is

$$i_f(t) = 40 - 40e^{-120t}$$

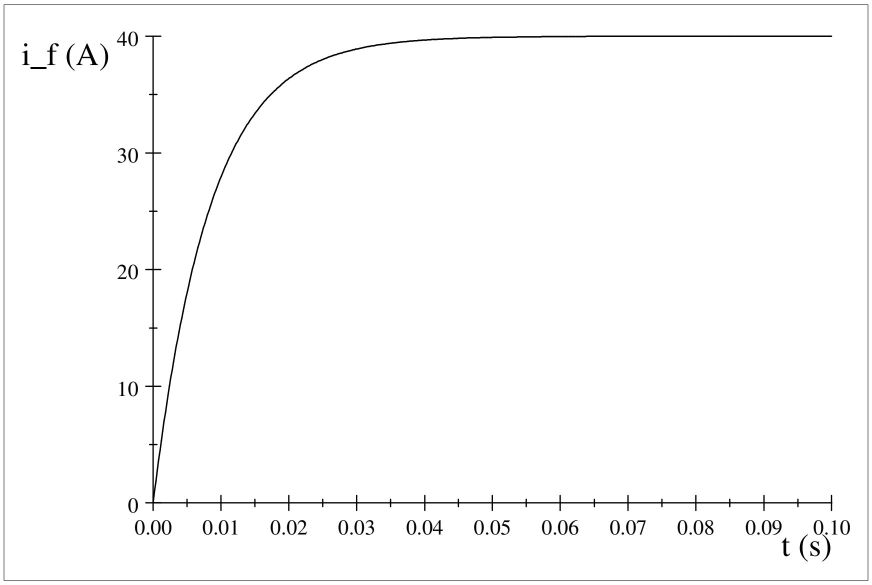


Figure3.4 The field current of the dc generator

The generator speed $\omega = \frac{2\pi n}{60} = \frac{2\pi\times 1500}{60} = 157 rad/s$. The armature current in s-domain is

$$
\begin{aligned}
I_a(s) &= \frac{K_e\omega(V_f(s) + L_f i_f(0)) + (L_a + L_L)i_a(0)(L_f s + R_f)}{(L_f s + R_f)((L_a + L_L)s + R_a + R_L)} \\
&= \frac{0.191 \times 157 \times (\frac{120}{s} + 0.025 \times 0) + (0.0025 + 0.04) \times 0 \times (0.025s + 3)}{(0.025s + 3)((0.0025 + 0.04)s + 0.2 + 40)} \\
&= \frac{0.191 \times 157 \times 120}{s(0.025s + 3)(0.0425s + 40.2)} \\
&= \frac{\frac{0.191\times 157\times 120}{0.025\times 0.0425}}{s\left(s + \frac{3}{0.025}\right)\left(s + \frac{40.2}{0.0425}\right)} \\
&= \frac{3.3868 \times 10^6}{s(s + 120.0)(s + 945.88)} \\
&= \frac{A}{s} + \frac{B}{s + 120} + \frac{C}{s + 945.88} \\
&= \frac{29.838}{s} + \frac{-34.174}{s + 120} + \frac{4.3355}{s + 945.88}
\end{aligned}
$$

where

$$A = s\frac{3.3868\times 10^6}{s(s+120.0)(s+945.88)}\Big|_{s=0} = \frac{3.3868\times 10^6}{(0+120)(0+945.88)} = 29.838$$

$$B = (s + 120)\frac{3.3868\times 10^6}{s(s+120.0)(s+945.88)}\Big|_{s=-120} = \frac{3.3868\times 10^6}{(-120)(-120+945.88)} = -34.174$$

$$C = (s + 945.88)\frac{3.3868\times 10^6}{s(s+120.0)(s+945.88)}\Big|_{s=-945.88} = \frac{3.3868\times 10^6}{(-945.88)(-945.88+120)} = 4.3355$$

Therefore, the field current in time domain is

$$i_a(t) = 29.838 - 34.174e^{-120t} + 4.3355e^{-945.88t}$$

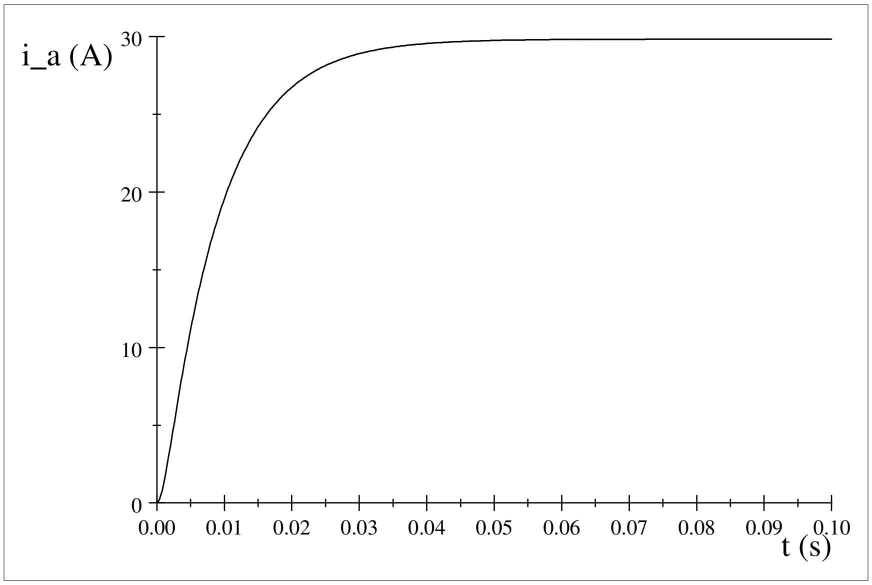


Figure3.5 The armature current of the dc generator

The induced voltage is given by

$$e_a(t) = K_e \omega i_f(t) = 0.191 \times 157 \times (40 - 40e^{-120t}) = 1199.5 - 1199.5e^{-120t}$$

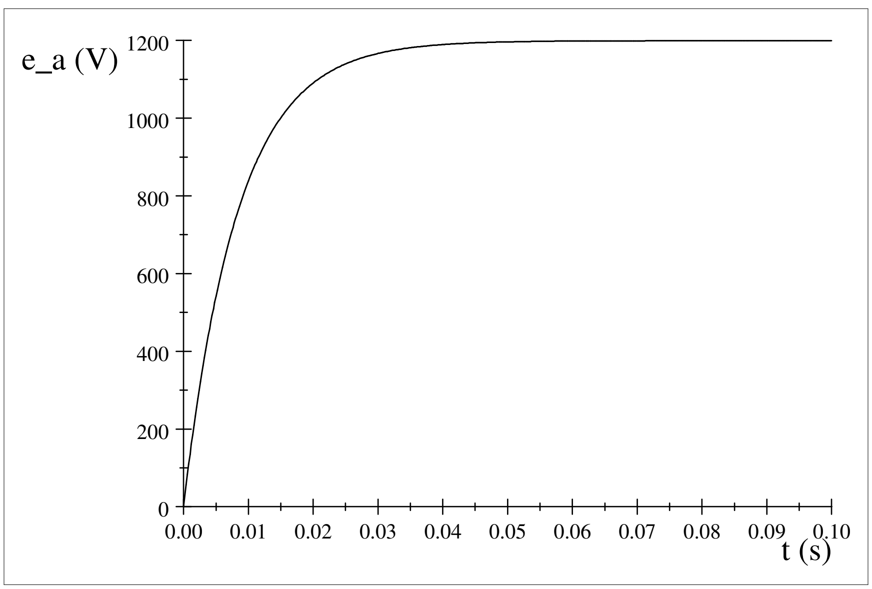


Figure3.6 The induced voltage of the dc generator

For practical purposes, the field current reaches its steady-state value after five time constant $5\tau_f = 5\frac{L_f}{R_f} = 5\frac{0.025}{3} = 0.042s$.

The final values of the field current, armature current, and induced voltage are $i_f(\infty) = 40A = \frac{V_f}{R_f}$, $i_a(\infty) = 29.838A = \frac{K_e \omega i_f(\infty)}{R_a + R_L}$, and $e_a(\infty) = 1199.5V = K_e \omega i_f(\infty)$.

3.3 DC Motor Dynamics

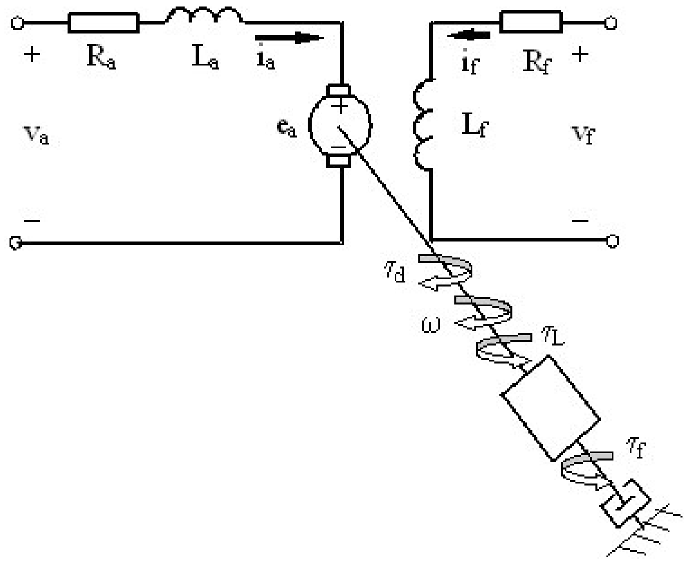


Figure3.7 DC motor dynamics

A dc motor is mainly composed of a stator, rotor, and commutator. The field winding is placed on the stator, which is also called the stator winding while the armature winding is mounted on the rotor, which is also referred to as the rotor winding. A pulsating induced voltage in the armature winding is converted to a dc voltage through the commutator. The equivalent circuit for a separately excited dc motor, together with a mechanical load, is shown in Figure 3.7.

For the field circuit, it follows from KVL that

$$v_f(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt}$$

where v_f, i_f, R_f, and L_f are the field voltage, current, resistance, and inductance, respectively.

For the armature circuit, according to KVL, we obtain

$$v_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + e_a(t)$$

where v_a, i_a, R_a, and L_a are the armature voltage, current, resistance, and inductance, respectively, and e_a is the back emf, which is determined by

$$e_a(t) = K_e i_f(t)\omega(t)$$

where K_e is the voltage constant and $\omega(t)$ is the angular speed of the motor.

For the mechnical load, it follows from Newton's law that

$$\tau_d(t) - \tau_L(t) - D\omega(t) = J\frac{d\omega(t)}{dt}$$

where D and J are the viscous friction coefficient and the moment of inertia of the rotating members, respectively, τ_L is the load torque and τ_d is the developed torque of the dc motor, which is determined by

$$\tau_d(t) = K_\tau i_f(t) i_a(t)$$

where K_τ is the torque constant, which is the same as the voltage constant K_e.

Substituting for e_a and τ_d in the three differential equations and solving them for the direvatives, it follows that

$$\frac{di_f(t)}{dt} = -\frac{R_f}{L_f} i_f(t) + \frac{1}{L_f} v_f(t)$$

$$\frac{di_a(t)}{dt} = -\frac{R_a}{L_a} i_a(t) - \frac{K_e}{L_a} i_f(t)\omega(t) + \frac{1}{L_a} v_a(t)$$

$$\frac{d\omega(t)}{dt} = \frac{K_e}{J} i_f(t) i_a(t) - \frac{1}{J}\tau_L(t) - \frac{D}{J}\omega(t)$$

which is a set of nonlinear differential equations. In these equations, both $v_f(t)$ and $v_a(t)$ can be adjusted to control the speed $\omega(t)$. When $v_f(t)$ is kept constant, that is, $i_f(t)$ is constant, the motor speed can be controlled by adjusting the armature voltage $v_a(t)$ and the motor is called the armature-controlled dc motor. On the other hand, when $v_a(t)$ is kept constant, the motor speed can be controlled by adjusting the field voltage $v_f(t)$ and the motor is called the field-controlled dc motor.

After the motor reaches the steady-state condition, $i_f(t)$, $i_a(t)$, and $\omega(t)$ remain constant, which implies that

$$\frac{di_f(t)}{dt} = 0, \frac{di_a(t)}{dt} = 0, \frac{d\omega(t)}{dt} = 0$$

Then, the following equations are obtained for the motor under steady-state condition.

$$v_f(\infty) = R_f i_f(\infty)$$
$$v_a(\infty) = R_a i_a(\infty) + e_a(\infty)$$
$$\tau_d(\infty) - \tau_L(\infty) - D\omega(\infty) = 0$$
$$e_a(\infty) = K_e i_f(\infty)\omega(\infty)$$
$$\tau_d(\infty) = K_e i_f(\infty) i_a(\infty)$$

or

$$0 = -R_f i_f(\infty) + v_f(\infty)$$
$$0 = -R_a i_a(\infty) - K_e i_f(\infty)\omega(\infty) + v_a(\infty)$$
$$0 = K_e i_f(\infty) i_a(\infty) - \tau_L(\infty) - D\omega(\infty)$$

from which one can determine the quantities $i_f(\infty)$, $i_a(\infty)$, and $\omega(\infty)$ under steady-state condition.

Example 3.3: A $240V$, $12hp$, separately excited dc motor has the following parameters $R_a = 0.28\Omega$, $L_a = 2.81mH$, $R_f = 320\Omega$, $L_f = 2H$, $K_e = 1.03$. $J = 0.087kg \cdot m$, and $D = 0.02N \cdot m \cdot s$. It is operating on a load of $15N \cdot m$ in the linear region of its magnetization characteristic. Determine the speed, field current, and armature current under steady-state condition.

Solution: The equations for the motor under steady-state condition are

$$0 = -R_f i_f(\infty) + v_f(\infty)$$
$$0 = -R_a i_a(\infty) - K_e i_f(\infty)\omega(\infty) + v_a(\infty)$$
$$0 = K_e i_f(\infty) i_a(\infty) - \tau_L(\infty) - D\omega(\infty)$$

Solving the first equation for $i_f(t)$ gives

$$i_f(\infty) = \frac{v_f(\infty)}{R_f} = \frac{240}{320} = 0.75A$$

Solving the second equation for $i_a(t)$ yields

$$i_a(\infty) = \frac{v_a(\infty)}{R_a} - \frac{K_e i_f(\infty)}{R_a}\omega(\infty)$$

Substituting this into the third equation produces

$$0 = K_e i_f(\infty)\left(\frac{v_a(\infty)}{R_a} - \frac{K_e i_f(\infty)}{R_a}\omega(\infty)\right) - \tau_L(\infty) - D\omega(\infty)$$

that is,

$$0 = K_e i_f(\infty) v_a(\infty) - K_e^2 i_f^2(\infty)\omega(\infty) - \tau_L(\infty) R_a - DR_a\omega(\infty)$$

Solving this for $\omega(\infty)$, we have

$$\omega(\infty) = \frac{K_e i_f(\infty) v_a(\infty) - \tau_L(\infty) R_a}{K_e^2 i_f^2(\infty) + DR_a} = \frac{1.03\times0.75\times240-15\times0.28}{1.03^2\times0.75^2+0.02\times0.28} = 300.82 rad/s$$

Therefore, the armature current is

$i_a(\infty) = \frac{v_a(\infty)}{R_a} - \frac{K_e i_f(\infty)}{R_a}\omega(\infty) = \frac{240}{0.28} - \frac{1.03\times0.75}{0.28} \times 300.82 = 27.202A$

3.4 Armature-Controlled DC Motors

For armature controlled dc motors, the field voltage is kept constant at V_f, so the field current is constant too, which implies that $\frac{di_f(t)}{dt} = 0$ and $i_f(t) = I_f = \frac{V_f}{R_f}$. The dynamic model for an armature controlled dc motor becomes

$$\frac{di_a(t)}{dt} = -\frac{R_a}{L_a}i_a(t) - \frac{K_e}{L_a}I_f\omega(t) + \frac{1}{L_a}v_a(t)$$
$$\frac{d\omega(t)}{dt} = \frac{K_e}{J}I_f i_a(t) - \frac{1}{J}\tau_L(t) - \frac{D}{J}\omega(t)$$

which is in the state-space form with state variables $i_a(t)$ and $\omega(t)$.
Taking the Laplace trasform, together with initial conditions $i_a(0)$ and $\omega(0)$, gives

$$sI_a(s) - i_a(0) = -\frac{R_a}{L_a}I_a(s) - \frac{K_e}{L_a}I_f\omega(s) + \frac{1}{L_a}V_a(s)$$
$$s\omega(s) - \omega(0) = \frac{K_e}{J}I_f I_a(s) - \frac{1}{J}\tau_L(s) - \frac{D}{J}\omega(s)$$

or

$$L_a sI_a(s) - L_a i_a(0) = -R_a I_a(s) - K_e I_f\omega(s) + V_a(s)$$
$$Js\omega(s) - J\omega(0) = K_e I_f I_a(s) - \tau_L(s) - D\omega(s)$$

Solving the first equation for $I_a(s)$ yields

$$I_a(s) = \frac{(V_a(s)+L_a i_a(0)) - K_e I_f\omega(s)}{L_a s + R_a}$$

Substituting into the second equation gives

$$Js\omega(s) - J\omega(0) = K_e I_f \frac{(V_a(s)+L_a i_a(0)) - K_e I_f\omega(s)}{L_a s + R_a} - \tau_L(s) - D\omega(s)$$

that is,

$$(Js + D)\omega(s) = \frac{K_e I_f(V_a(s)+L_a i_a(0))}{L_a s + R_a} - \frac{(K_e I_f)^2\omega(s)}{L_a s + R_a} + J\omega(0) - \tau_L(s)$$

Solving this for $\omega(s)$ produces

$$\omega(s) = \frac{K_e I_f(V_a(s)+L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js+D)(L_a s + R_a) + (K_e I_f)^2}$$

Then, the armature current is given by

$$
\begin{aligned}
I_a(s) &= \frac{(V_a(s)+L_a i_a(0)) - K_e I_f \omega(s)}{L_a s + R_a} \\
&= \frac{(V_a(s)+L_a i_a(0)) - K_e I_f \frac{K_e I_f (V_a(s)+L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js+D)(L_a s + R_a) + (K_e I_f)^2}}{L_a s + R_a} \\
&= \frac{\frac{(V_a(s)+L_a i_a(0))\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right)}{(Js+D)(L_a s+R_a)+(K_e I_f)^2} - K_e I_f \frac{K_e I_f (V_a(s)+L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js+D)(L_a s + R_a) + (K_e I_f)^2}}{L_a s + R_a} \\
&= \frac{\begin{array}{c}(V_a(s)+L_a i_a(0))\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right) \\ -K_e I_f (K_e I_f (V_a(s)+L_a i_a(0)) + (L_a s+R_a)(J\omega(0) - \tau_L(s)))\end{array}}{(L_a s+R_a)\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right)} \\
&= \frac{\begin{array}{c}(V_a(s)+L_a i_a(0))(Js+D)(L_a s+R_a) + (K_e I_f)^2 (V_a(s)+L_a i_a(0)) \\ -(K_e I_f)^2 (V_a(s)+L_a i_a(0)) - K_e I_f (L_a s+R_a)(J\omega(0) - \tau_L(s))\end{array}}{(L_a s+R_a)\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right)} \\
&= \frac{(V_a(s)+L_a i_a(0))(Js+D)(L_a s+R_a) - K_e I_f (L_a s+R_a)(J\omega(0) - \tau_L(s))}{(L_a s+R_a)\left((Js+D)(L_a s+R_a)+(K_e I_f)^2\right)} \\
&= \frac{(V_a(s)+L_a i_a(0))(Js+D) - K_e I_f (J\omega(0) - \tau_L(s))}{(Js+D)(L_a s+R_a)+(K_e I_f)^2}
\end{aligned}
$$

Example 3.4: (see Example 3.3) A $240V$, $12hp$, separately excited dc motor has the following parameters $R_a = 0.28\Omega$, $L_a = 2.81mH$, $R_f = 320\Omega$, $L_f = 2H$, $K_e = 1.03$. $J = 0.087kg \cdot m$, and $D = 0.02N \cdot m \cdot s$. Determine its speed and armature current as a function of time when it is suddenly connected to a 240V dc source at no load condition.

Solution: Prior to the application of armature voltage the motor speed and armature current are zero. That is, at $t = 0$, $i_a(0) = 0$ and $\omega(0) = 0$. In addition, the load torque is zero because the motor operates at no load. That is, $\tau_L(t) = 0$. The field current is

$$I_f = \frac{V_f}{R_f} = \frac{240}{320} = 0.75A$$

Note that the armature voltage $v_a(t)$ is a step signal with amplitude of 240V, so its Laplace transform is $V_a(s) = \frac{240}{s}$.

Therefore, we have

$$
\begin{aligned}
\omega(s) &= \frac{K_e I_f (V_a(s) + L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js + D)(L_a s + R_a) + (K_e I_f)^2} \\
&= \frac{1.03 \times 0.75 \times \frac{240}{s}}{(0.087s + 0.02)(0.00281s + 0.28) + (1.03 \times 0.75)^2} \\
&= \frac{185.4}{s(2.4447 \times 10^{-4} s^2 + 2.4416 \times 10^{-2} s + 0.60236)} \\
&= \frac{\frac{185.4}{2.4447 \times 10^{-4}}}{s\left(s^2 + \frac{2.4416 \times 10^{-2}}{2.4447 \times 10^{-4}} s + \frac{0.60236}{2.4447 \times 10^{-4}}\right)} \\
&= \frac{7.5838 \times 10^5}{s(s^2 + 99.873s + 2463.9)} \\
&= \frac{7.5838 \times 10^5}{s(s + 44.482)(s + 55.391)}
\end{aligned}
$$

In order to determine the inverse Laplace transform, we expand $\omega(s)$ into partial fractions as

$$\omega(s) = \frac{A}{s} + \frac{B}{s+44.482} + \frac{C}{s+55.391}$$

where A, B, and Ccan now be determined by the root-substitution method. Thus,

$$A = s\frac{7.5838 \times 10^5}{s(s+44.482)(s+55.391)}\bigg|_{s=0} = \frac{7.5838 \times 10^5}{(0+44.482)(0+55.391)} = 307.80$$

$$B = (s + 44.482)\frac{7.5838 \times 10^5}{s(s+44.482)(s+55.391)}\bigg|_{s=-44.482} = \frac{7.5838 \times 10^5}{(-44.482)(-44.482+55.391)} = -1562.9$$

$$C = (s + 55.391)\frac{7.5838 \times 10^5}{s(s+44.482)(s+55.391)}\bigg|_{s=-55.391} = \frac{7.5838 \times 10^5}{(-55.391)(-55.391+44.482)} = 1255.1$$

Finally, we can take the inverse Laplace transform of

$$\omega(s) = \frac{307.80}{s} + \frac{-1562.9}{s+44.482} + \frac{1255.1}{s+55.391}$$

and get the angular velocity as

$$\omega(t) = 307.80 - 1562.9e^{-44.482t} + 1255.1e^{-55.391t}$$

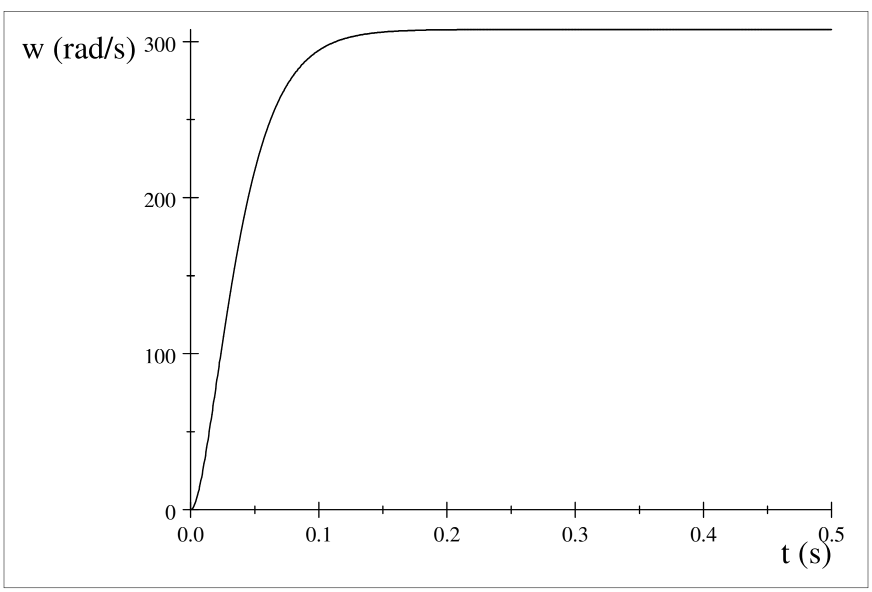


Figure3.8 The motor speed

The Laplace transform of the armature current is

$$
\begin{aligned}
I_a(s) &= \frac{(V_a(s)+L_a i_a(0))(Js+D)-K_e I_f(J\omega(0)-\tau_L(s))}{(Js+D)(L_a s+R_a)+(K_e I_f)^2} \\
&= \frac{\frac{240}{s}(0.087s+0.02)}{(0.087s+0.02)(0.00281s+0.28)+(1.03\times 0.75)^2} \\
&= \frac{240(0.087s+0.02)}{s(2.4447\times 10^{-4}s^2+2.4416\times 10^{-2}s+0.60236)} \\
&= \frac{\frac{240\times 0.087}{2.4447\times 10^{-4}}s+\frac{240\times 0.02}{2.4447\times 10^{-4}}}{s\left(s^2+\frac{2.4416\times 10^{-2}}{2.4447\times 10^{-4}}s+\frac{0.60236}{2.4447\times 10^{-4}}\right)} \\
&= \frac{85409s+19634}{s(s^2+99.873s+2463.9)} \\
&= \frac{85409s+19634}{s(s+44.482)(s+55.391)}
\end{aligned}
$$

In terms of its partial fraction expansion, $I_a(s)$ can be written as

$$i_a(s) = \frac{A}{s}+\frac{B}{s+44.482}+\frac{C}{s+55.391} = \frac{7.9687}{s}+\frac{7788.8}{s+44.482}+\frac{-7796.7}{s+55.391}$$

where

$$A = s\frac{85409s+19634}{s(s+44.482)(s+55.391)}\bigg|_{s=0} = \frac{85409\times 0+19634}{(0+44.482)(0+55.391)} = 7.9687$$

$$B = (s+44.482)\frac{85409s+19634}{s(s+44.482)(s+55.391)}\bigg|_{s=-44.482} = \frac{85409\times(-44.482)+19634}{(-44.482)(-44.482+55.391)} = 7788.8$$

$$C = (s+55.391)\frac{85409s+19634}{s(s+44.482)(s+55.391)}\bigg|_{s=-55.391} = \frac{85409\times(-55.391)+19634}{(-55.391)(-55.391+44.482)} = -7796.7$$

Finally, we obtain the armature current as

$$i_a(t) = 7.9687+7788.8e^{-44.482t}-7796.7e^{-55.391t}$$

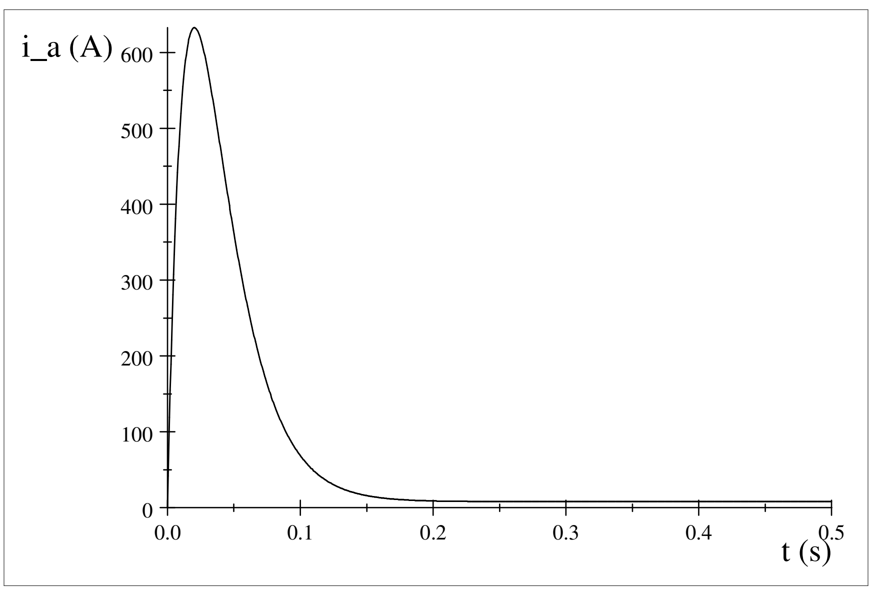


Figure3.9 The motor armature current

The rated current is $I_a(rating) = \frac{12\times 746}{240} = 37.3A$. The starting current is way too high so that the motor will be burnt.

Note that the mechanical time constant is $\tau_m = \frac{J}{D} = \frac{0.087}{0.02} = 4.35s$ and the electrical time constant is

$\tau_e = \frac{L_a}{R_a} = \frac{0.00281}{0.28} = 1.0036 \times 10^{-2} s.$

Example 3.5: (See Example 3.3) A $240V$, $12hp$, separately excited dc motor has the following parameters $R_a = 0.28\Omega$, $L_a = 2.81mH$, $R_f = 320\Omega$, $L_f = 2H$, $K_e = 1.03$. $J = 0.087kg \cdot m$, and $D = 0.02N \cdot m \cdot s$. Determine its speed and armature current as a function of time when it is suddenly connected to a $30V$ dc source at a load of $15N \cdot m$.

Solution: Prior to the application of armature voltage the motor speed and armature current are zero. That is, at $t = 0$, $i_a(0) = 0$ and $\omega(0) = 0$. In addition, the load torque is $15N \cdot m$, that is, $\tau_L(t) = 15$. The field current is

$I_f = \frac{V_f}{R_f} = \frac{240}{320} = 0.75A$

Note that $V_a(s) = \frac{50}{s}$ and $\tau_L(s) = \frac{15}{s}$.

Therefore, we have

$$\begin{aligned}
\omega(s) &= \frac{K_e I_f (V_a(s) + L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js + D)(L_a s + R_a) + (K_e I_f)^2} \\
&= \frac{1.03 \times 0.75 \times \frac{50}{s} + (0.00281s + 0.28)(-\frac{15}{s})}{(0.087s + 0.02)(0.00281s + 0.28) + (1.03 \times 0.75)^2} \\
&= \frac{1.03 \times 0.75 \times 50 + (0.00281s + 0.28)(-15)}{s(2.4447 \times 10^{-4} s^2 + 2.4416 \times 10^{-2} s + 0.60236)} \\
&= \frac{34.425 - 0.04215s}{s(2.4447 \times 10^{-4} s^2 + 2.4416 \times 10^{-2} s + 0.60236)} \\
&= \frac{\frac{34.425}{2.4447 \times 10^{-4}} - \frac{0.04215}{2.4447 \times 10^{-4}} s}{s\left(s^2 + \frac{2.4416 \times 10^{-2}}{2.4447 \times 10^{-4}} s + \frac{0.60236}{2.4447 \times 10^{-4}}\right)} \\
&= \frac{1.4081 \times 10^5 - 172.41s}{s(s^2 + 99.873s + 2463.9)} \\
&= \frac{1.4081 \times 10^5 - 172.41s}{s(s + 44.482)(s + 55.391)}
\end{aligned}$$

In order to determine the inverse Laplace transform, we expand $\omega(s)$ into partial fractions as

$\omega(s) = \frac{A}{s} + \frac{B}{s+44.482} + \frac{C}{s+55.391}$

where A, B, and C can now be determined by the root-substitution method. Thus,

$A = s\frac{1.4081\times10^5 - 172.41s}{s(s+44.482)(s+55.391)}\Big|_{s=0} = \frac{1.4081\times10^5 - 172.41\times0}{(0+44.482)(0+55.391)} = 57.149$

$B = (s + 44.482)\frac{1.4081\times10^5 - 172.41s}{s(s+44.482)(s+55.391)}\Big|_{s=-44.482} = \frac{1.4081\times10^5 - 172.41\times(-44.482)}{(-44.482)(-44.482+55.391)} = -305.98$

$C = (s + 55.391)\frac{1.4081\times10^5 - 172.41s}{s(s+44.482)(s+55.391)}\Big|_{s=-55.391} = \frac{1.4081\times10^5 - 172.41\times(-55.391)}{(-55.391+44.482)(-55.391)} = 248.83$

Finally, we can take the inverse Laplace transform of

$\omega(s) = \frac{57.149}{s} + \frac{-305.98}{s+44.482} + \frac{248.83}{s+55.391}$

and get the angular velocity as

$\omega(t) = 57.149 - 305.98e^{-44.482t} + 248.83e^{-55.391t}$

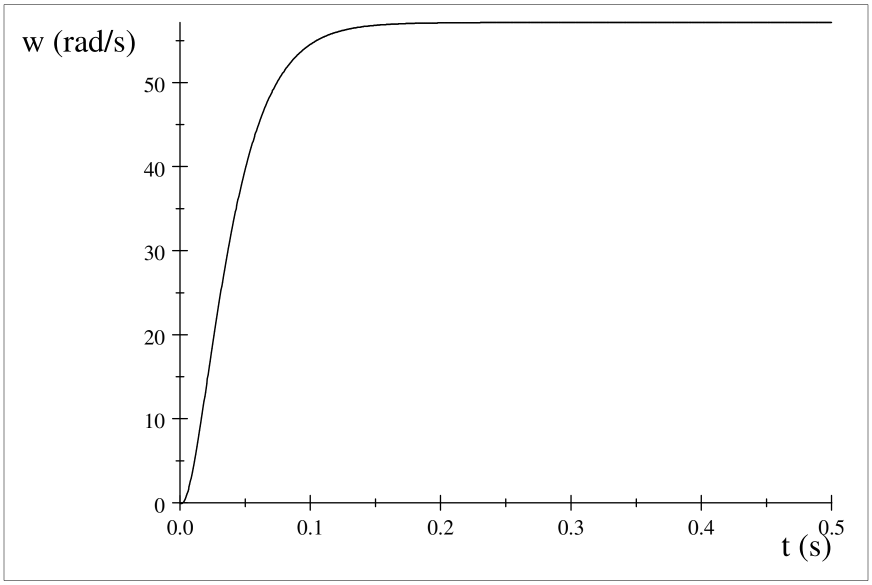


Figure3.10 The motor speed

The Laplace transform of the armature current is

$$I_a(s) = \frac{(V_a(s) + L_a i_a(0))(Js + D) - K_e I_f(J\omega(0) - \tau_L(s))}{(Js + D)(L_a s + R_a) + (K_e I_f)^2}$$

$$= \frac{\frac{30}{s}(0.087s + 0.02) - 1.03 \times 0.75(-\frac{15}{s})}{(0.087s + 0.02)(0.00281s + 0.28) + (1.03 \times 0.75)^2}$$

$$= \frac{30(0.087s + 0.02) - 1.03 \times 0.75(-15)}{s(2.4447 \times 10^{-4}s^2 + 2.4416 \times 10^{-2}s + 0.60236)}$$

$$= \frac{2.61s + 12.188}{s(2.4447 \times 10^{-4}s^2 + 2.4416 \times 10^{-2}s + 0.60236)}$$

$$= \frac{\frac{2.61}{2.4447\times10^{-4}}s + \frac{12.188}{2.4447\times10^{-4}}}{s\left(s^2 + \frac{2.4416\times10^{-2}}{2.4447\times10^{-4}}s + \frac{0.60236}{2.4447\times10^{-4}}\right)}$$

$$= \frac{10676s + 49855}{s(s^2 + 99.873s + 2463.9)}$$

$$= \frac{10676s + 49855}{s(s + 44.482)(s + 55.391)}$$

In terms of its partial fraction expansion, $I_a(s)$ can be written as

$$\omega(s) = \frac{A}{s} + \frac{B}{s+44.482} + \frac{C}{s+55.391} = \frac{20.234}{s} + \frac{875.9}{s+44.482} + \frac{-896.14}{s+55.391}$$

where

$$A = s\frac{10676s+49855}{s(s+44.482)(s+55.391)}\bigg|_{s=0} = \frac{10676\times0+49855}{(0+44.482)(0+55.391)} = 20.234$$

$$B = (s + 44.482)\frac{10676s+49855}{s(s+44.482)(s+55.391)}\bigg|_{s=-44.482} = \frac{10676\times(-44.482)+49855}{(-44.482)(-44.482+55.391)} = 875.9$$

$$C = (s + 55.391)\frac{10676s+49855}{s(s+44.482)(s+55.391)}\bigg|_{s=-55.391} = \frac{10676\times(-55.391)+49855}{(-55.391+44.482)(-55.391)} = -896.14$$

Finally, we obtain the armature current as

$$i_a(t) = 20.234 + 875.9e^{-44.482t} - 896.14e^{-55.391t}$$

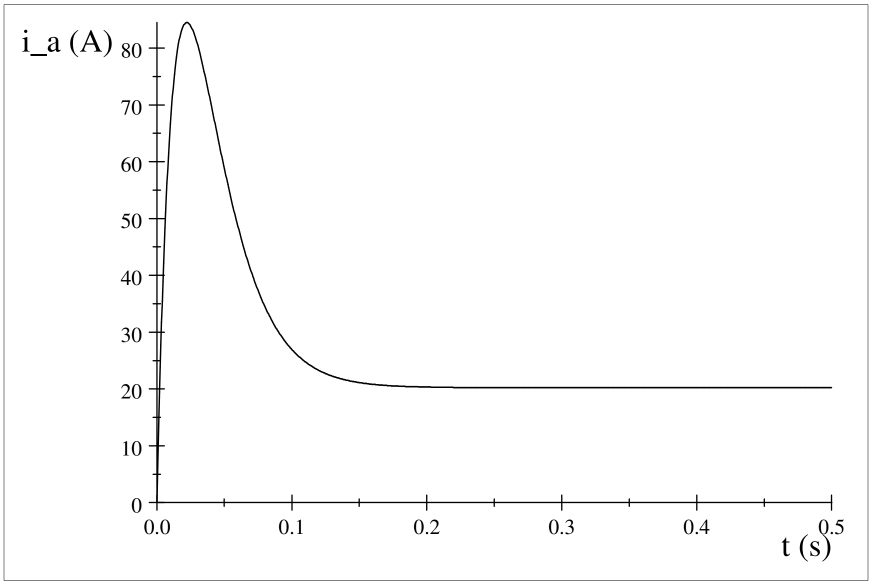


Figure3.11 The motor armature current

3.5 Field-Controlled DC Motors

In an armature-controlled dc motor, the field current is kept at a constant level and the armature voltage is adjusted to vary the speed below its rated speed. In a field-controlled dc motor, however, we will change the field current in order to obtain a motor speed higher than its rated speed.

The mathmatical model for a field-controlled dc motor is given below.

$$\frac{di_f(t)}{dt} = -\frac{R_f}{L_f} i_f(t) + \frac{1}{L_f} v_f(t)$$
$$\frac{di_a(t)}{dt} = -\frac{R_a}{L_a} i_a(t) - \frac{K_e}{L_a} i_f(t)\omega(t) + \frac{V_a}{L_a}$$
$$\frac{d\omega(t)}{dt} = \frac{K_e}{J} i_f(t) i_a(t) - \frac{1}{J}\tau_L(t) - \frac{D}{J}\omega(t)$$

It is clear that these equations are nonlinear because of the products of the state variables in these equations. As a result, the Laplace transform approach would not be appropriate to get closed-form solutions for $i_f(t)$, $i_a(t)$ and $\omega(t)$. However, a simplifying assumption can be made to linearize these equations.

In an electric motor, the time constant of the electric circuit is much smaller than the time constant of the mechnical parts. Therefore, it can be considered that the time constant of the field circuit is much smaller than the mechanical time constant of the motor. The field current reaches its steady-state before the armature responds to the changes in the field current. Therefore, we have

$$\frac{di_f(t)}{dt} = -\frac{R_f}{L_f} i_f(t) + \frac{1}{L_f} v_f(t)$$
$$\frac{di_a(t)}{dt} = -\frac{R_a}{L_a} i_a(t) - \frac{K_e}{L_a} I_f\omega(t) + \frac{V_a}{L_a}$$
$$\frac{d\omega(t)}{dt} = \frac{K_e}{J} I_f i_a(t) - \frac{1}{J}\tau_L(t) - \frac{D}{J}\omega(t)$$

Taking the Laplace transform gives

$$sI_f(s) - i_f(0) = -\frac{R_f}{L_f}I_f(s) + \frac{1}{L_f}V_f(s)$$
$$sI_a(s) - i_a(0) = -\frac{R_a}{L_a}I_a(s) - \frac{K_e}{L_a}I_f\omega(s) + \frac{1}{L_a}V_a(s)$$
$$s\omega(s) - \omega(0) = \frac{K_e}{J}I_fI_a(s) - \frac{1}{J}\tau_L(s) - \frac{D}{J}\omega(s)$$

or

$$L_fsI_f(s) - L_fi_f(0) = -R_fI_f(s) + V_f(s)$$
$$L_asI_a(s) - L_ai_a(0) = -R_aI_a(s) - K_eI_f\omega(s) + V_a(s)$$
$$Js\omega(s) - J\omega(0) = K_eI_fI_a(s) - \tau_L(s) - D\omega(s)$$

Solving these equations yields

$$I_f(s) = \frac{V_f(s)+L_fi_f(0)}{L_fs+R_f}$$
$$\omega(s) = \frac{K_eI_f(V_a(s)+L_ai_a(0))+(L_as+R_a)(J\omega(0)-\tau_L(s))}{(Js+D)(L_as+R_a)+(K_eI_f)^2}$$
$$I_a(s) = \frac{(V_a(s)+L_ai_a(0))(Js+D)-K_eI_f(J\omega(0)-\tau_L(s))}{(Js+D)(L_as+R_a)+(K_eI_f)^2}$$

Example 3.6: (See Example 3.3) A $240V$, $12hp$, separately excited dc motor has the following parameters $R_a = 0.28\Omega$, $L_a = 2.81mH$, $R_f = 320\Omega$, $L_f = 2H$, $K_e = 1.03$. $J = 0.087kg \cdot m$, and $D = 0.02N \cdot m \cdot s$. It is operating on a load of $15N \cdot m$ in the linear region of its magnetization characteristic. Determine its speed, field current, and armature current as a function of time when the field voltage is suddenly reduced from $240V$ to $192V$ at $t = 0$.

Solution: Since the motor has already been operating at steady state on a load of $\tau_L = 15N \cdot m$ before the field voltage is suddenly changed, we have to evaluate the initial conditions on $i_f(t)$, $i_a(t)$ and $\omega(t)$ from the equations for the steady-state operation, which is done in Example 3.2 and the initial values are

$$i_f(0) = 0.75A, \omega(0) = 300.82rad/s, i_a(0) = 27.202A$$

First, we will determine the field current as follows:

$$I_f(s) = \frac{V_f(s)+L_fi_f(0)}{L_fs+R_f} = \frac{V_f(s)+L_fi_f(0)}{L_fs+R_f} = \frac{\frac{192}{s}+2\times0.75}{2s+320} = \frac{96+0.75s}{s(s+160)} = \frac{A}{s} + \frac{B}{s+160} = \frac{0.6}{s} + \frac{0.15}{s+160}$$

where

$$A = s\frac{96+0.75s}{s(s+160)}\bigg|_{s=0} = \frac{96+0.75\times0}{(0+160)} = 0.6$$
$$B = s\frac{96+0.75s}{s(s+160)}\bigg|_{s=-160} = \frac{96+0.75\times(-160)}{-160} = 0.15$$

Taking the inverse Laplace transform produces

$$i_f(t) = 0.6 + 0.15e^{-160t}$$

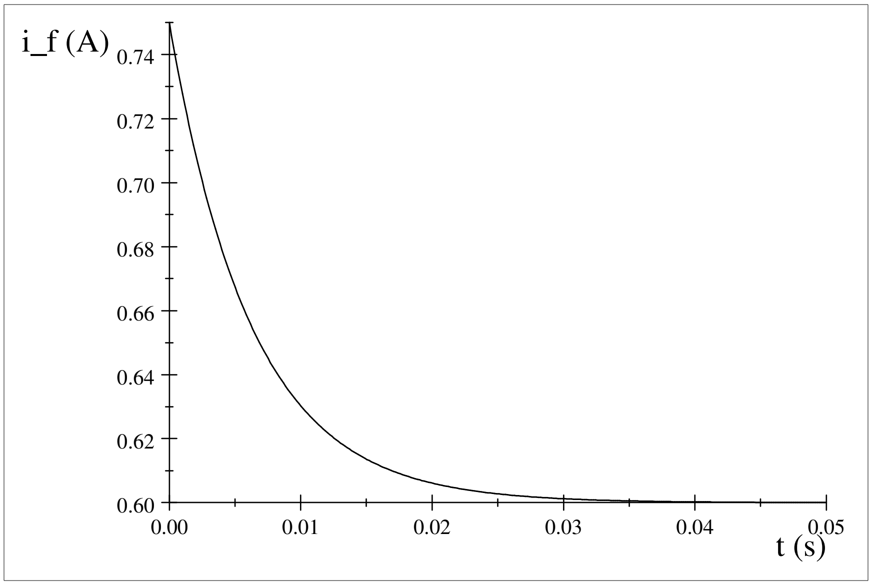


Figure3.12 The motor field current

which has a steady state value $I_f = 0.6A$.

For the motor speed, we have

$$
\begin{aligned}
\omega(s) &= \frac{K_e I_f (V_a(s) + L_a i_a(0)) + (L_a s + R_a)(J\omega(0) - \tau_L(s))}{(Js + D)(L_a s + R_a) + (K_e I_f)^2} \\
&= \frac{1.03 \times 0.6 \times (\frac{240}{s} + 0.00281 \times 27.2) + (0.00281s + 0.28)(0.087 \times 300.79 - \frac{15}{s})}{(0.087s + 0.02)(0.00281s + 0.28) + (1.03 \times 0.6)^2} \\
&= \frac{1.03 \times 0.6 \times (240 + 0.00281 \times 27.2s) + (0.00281s + 0.28)(0.087 \times 300.79s - 15)}{s(2.4447 \times 10^{-4}s^2 + 2.4416 \times 10^{-2}s + 0.38752)} \\
&= \frac{7.3534 \times 10^{-2}s^2 + 7.3323s + 144.12}{s(2.4447 \times 10^{-4}s^2 + 2.4416 \times 10^{-2}s + 0.38752)} \\
&= \frac{\frac{7.3534\times10^{-2}}{2.4447\times10^{-4}}s^2 + \frac{7.3323}{2.4447\times10^{-4}}s + \frac{144.12}{2.4447\times10^{-4}}}{s\left(s^2 + \frac{2.4416\times10^{-2}}{2.4447\times10^{-4}}s + \frac{0.38752}{2.4447\times10^{-4}}\right)} \\
&= \frac{300.79s^2 + 29993.s + 5.8952 \times 10^5}{s(s^2 + 99.873s + 1585.1)} \\
&= \frac{300.79s^2 + 29993.s + 5.8952 \times 10^5}{s(s + 19.794)(s + 80.079)} \\
&= \frac{A}{s} + \frac{B}{s + 19.794} + \frac{C}{s + 80.079} \\
&= \frac{371.92}{s} + \frac{-95.274}{s + 19.794} + \frac{24.147}{s + 80.079}
\end{aligned}
$$

where A, B, and C are determined by

$$A = s\frac{300.79s^2+29993s+5.8952\times10^5}{s(s+19.794)(s+80.079)}\Big|_{s=0} = \frac{300.79\times0^2+29993\times0+5.8952\times10^5}{(0+19.794)(0+80.079)} = 371.92$$

$$B = (s + 19.794)\frac{300.79s^2+29993s+5.8952\times10^5}{s(s+19.794)(s+80.079)}\Big|_{s=-19.794} = \frac{300.79\times(-19.794)^2+29993\times(-19.794)+5.8952\times10^5}{(-19.794)(-19.794+80.079)} =$$

-95.274

$$C = (s+80.079)\frac{300.79s^2+29993s+5.8952\times10^5}{s(s+19.794)(s+80.079)}\Bigg|_{s=-80.079} = \frac{300.79\times(-80.079)^2+29993\times(-80.079)+5.8952\times10^5}{(-80.079)(-80.079+19.794)} = 24.147$$

Finally, we can take the inverse Laplace transform to get the angular velocity as

$$\omega(t) = 371.92 - 95.274e^{-19.794t} + 24.147e^{-80.079t}$$

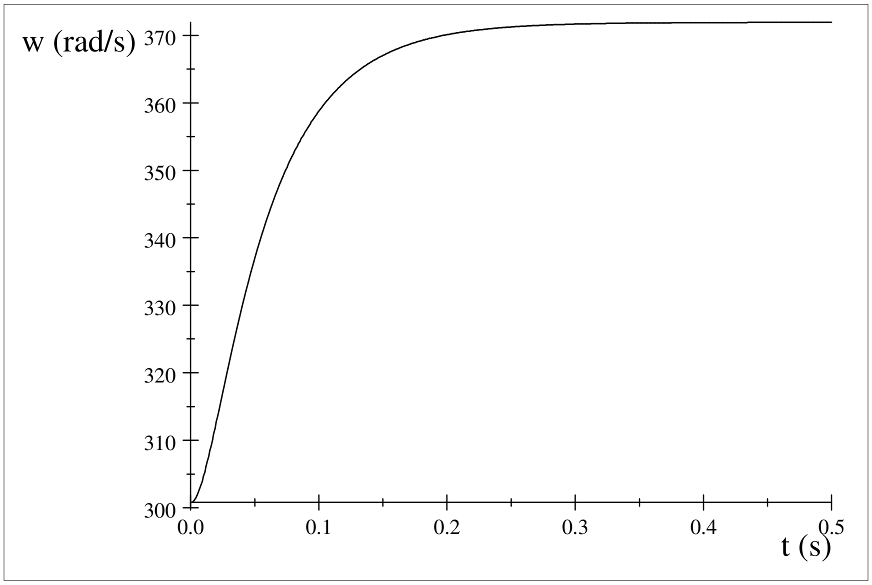


Figure3.13 The motor speed

The Laplace transform of the armature current is

$$I_a(s) = \frac{(V_a(s)+L_a i_a(0))(Js+D) - K_e I_f(J\omega(0) - \tau_L(s))}{(Js+D)(L_a s+R_a) + (K_e I_f)^2}$$

$$= \frac{(\frac{240}{s} + 0.00281\times27.2)(0.087s+0.02) - 1.03\times0.6(0.087\times300.79 - \frac{15}{s})}{(0.087s+0.02)(0.00281s+0.28) + (1.03\times0.6)^2}$$

$$= \frac{(240+0.00281\times27.2s)(0.087s+0.02) - 1.03\times0.6(0.087\times300.79s - 15)}{s(2.4447\times10^{-4}s^2 + 2.4416\times10^{-2}s + 0.38752)}$$

$$= \frac{6.6496\times10^{-3}s^2 + 4.7093s + 14.07}{s(2.4447\times10^{-4}s^2 + 2.4416\times10^{-2}s + 0.38752)}$$

$$= \frac{\frac{6.6496\times10^{-3}}{2.4447\times10^{-4}}s^2 + \frac{4.7093}{2.4447\times10^{-4}}s + \frac{14.07}{2.4447\times10^{-4}}}{s\left(s^2 + \frac{2.4416\times10^{-2}}{2.4447\times10^{-4}}s + \frac{0.38752}{2.4447\times10^{-4}}\right)}$$

$$= \frac{27.2s^2 + 19263s + 57553}{s(s^2+99.873s+1585.1)}$$

$$= \frac{27.2s^2 + 19263s + 57553}{s(s+19.794)(s+80.079)}$$

$$= \frac{A}{s} + \frac{B}{s+19.794} + \frac{C}{s+80.079}$$

$$= \frac{36.309}{s} + \frac{262.37}{s+19.794} + \frac{-271.48}{s+80.079}$$

where

$$A = s\frac{27.2s^2+19263s+57553}{s(s+19.794)(s+80.079)}\bigg|_{s=0} = \frac{27.2\times(0)^2+19263\times(0)+57553}{(0+19.794)(0+80.079)} = 36.309$$

$$B = (s+19.794)\frac{27.2s^2+19263s+57553}{s(s+19.794)(s+80.079)}\bigg|_{s=-19.794} = \frac{27.2\times(-19.794)^2+19263\times(-19.794)+57553}{(-19.794)(-19.794+80.079)} = 262.37$$

$$C = (s+80.079)\frac{27.2s^2+19263s+57553}{s(s+19.794)(s+80.079)}\bigg|_{s=-80.079} = \frac{27.2\times(-80.079)^2+19263\times(-80.079)+57553}{(-80.079)(-80.079+19.794)} = -271.48$$

Finally, we obtain the armature current as

$$i_a(t) = 36.309 + 262.37e^{-19.794t} - 271.48e^{-80.079t}$$

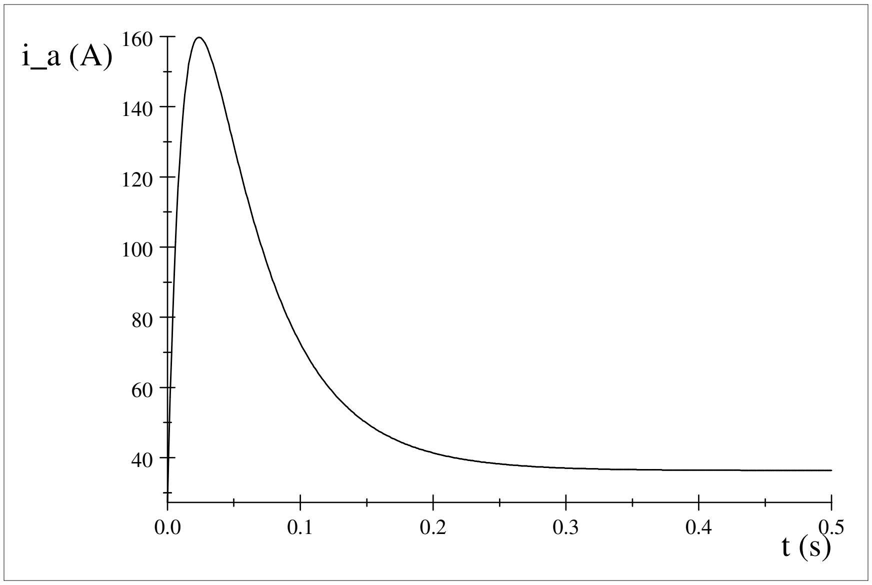


Figure3.14 The motor armature current

It is clear that the field current reaches its steady state at about 30ms whereas it takes about 300ms for the speed and thereby the armature current to do so. This is consistent with our assumption that the mechanical response is much slower than the changes in the field current.

It is important to note that the armature current reaches its peak at 160A, which is well over its rated value. This is mainly caused by the large mechanical time constant of the motor that does not allow a rapid change in the back emf of the motor. Therefore, it is recommended that the field current be gradualy varied so that high currents will not take place in the armature circuit.

Chapter 4 Transformers

4.1 Ideal Transformers

Figure 4.1 shows a transformer circuit.

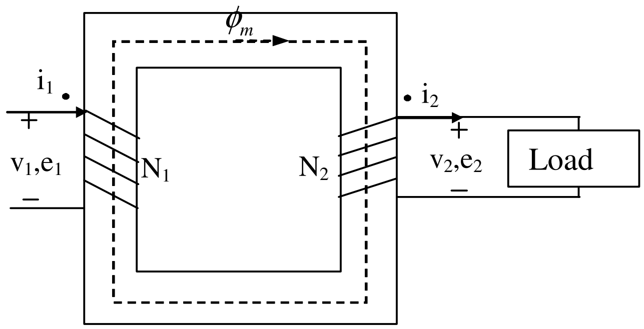


Figure 4.1 An ideal transformer

The dot markings indicate terminals of corresponding polarity, that is, if one follows through the primary and secondary windings beginning at their marked terminals, one will find that both windings encircle the core in the same direction with respect to the flux. Therefore, if one compares the voltages of the two windings, the voltages from the dot-marked to an unmarked terminal will have the same instantaneous polarity for both windings.

A transformer is called the ideal transformer if the following assumptions are satisfied:

(A1) The core of the transformer is highly permeable so that is requires vanishingly small magnetomotive force (mmf) to set up the flux ϕ.

(A2) There is no eddy-current or hysteresis loss.

(A3) There is no resistance.

(A4) There is no leakage flux.

With these assumptions, it is obvious that

$$v_1 = e_1 = \frac{d\lambda_1}{dt} = \frac{d(N_1\phi)}{dt} = N_1\frac{d\phi}{dt}$$

$$v_2 = e_2 = \frac{d\lambda_2}{dt} = \frac{d(N_2\phi)}{dt} = N_2\frac{d\phi}{dt}$$

which implies that

$$\frac{v_1}{v_2} = \frac{e_1}{e_2} = \frac{N_1}{N_2} = a$$

where a is referred to as turns ratio or transformation ratio.

Since there is no loss in the ideal transformer, the input power is the same as the output power, that is,

$$v_1 i_1 = v_2 i_2$$

As a result, we have

$$\frac{i_2}{i_1} = \frac{v_1}{v_2} = a$$

Now suppose that the instantaneous flux is $\phi = \phi_{\max}\sin(\omega t)$. Then we have

$$v_1 = e_1 = N_1\frac{d\phi}{dt} = N_1\omega\phi_{\max}\cos(\omega t) = \sqrt{2}\,V_1\cos(\omega t)$$

$$v_2 = e_2 = N_2\frac{d\phi}{dt} = N_2\omega\phi_{\max}\cos(\omega t) = \sqrt{2}\,V_2\cos(\omega t)$$

where $V_1 = \frac{N_1\omega\phi_{\max}}{\sqrt{2}}$ and $V_2 = \frac{N_2\omega\phi_{\max}}{\sqrt{2}}$ are rms values of v_1 and v_2.

It is a common practice to express sinusoidal signals $i_1, i_2, e_1, e_2, v_1,$ and v_2 in terms of phasors as $\hat{I}_1, \hat{I}_2, \hat{E}_1, \hat{E}_2, \hat{V}_1,$ and $\hat{V}_2$. Then, we have

$$\frac{\hat{I}_2}{\hat{I}_1} = \frac{\hat{E}_1}{\hat{E}_2} = \frac{\hat{V}_1}{\hat{V}_2} = a$$

If $\hat{Z}_2$ is the load impedance on the secondary side, then

$$\hat{Z}_2 = \frac{\hat{V}_2}{\hat{I}_2} = \frac{\frac{\hat{V}_1}{a}}{a\hat{I}_1} = \frac{1}{a^2}\frac{\hat{V}_1}{\hat{I}_1} = \frac{1}{a^2}\hat{Z}_1$$

where $\hat{Z}_1 = \frac{\hat{V}_1}{\hat{I}_1}$ is the load impedance as referred to the primary side. The equivalent circuit for an ideal transformer is shown in Figure 4.2.

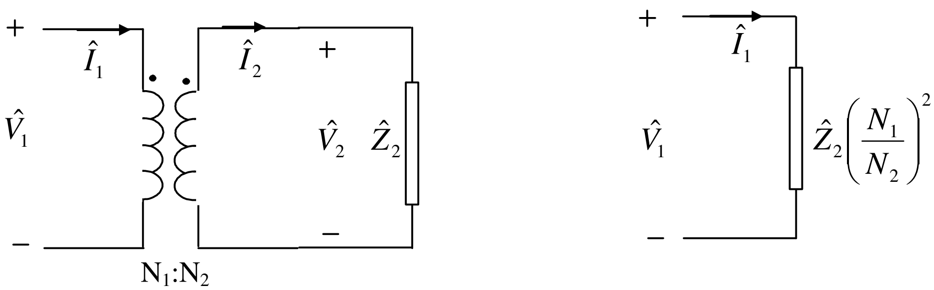


Figure 4.2 The equivalent circuit of an ideal transformer

4.2 Practical Transformers

For a practical transformer, both primary and secondary windings have resistances, denoted R_1 and R_2, and leakage fluxes, denoted ϕ_{l1} and ϕ_{l2} as shown in Figure 4.2, which link their own windings through air and can be modelled by leakage reactances X_1 and X_2.

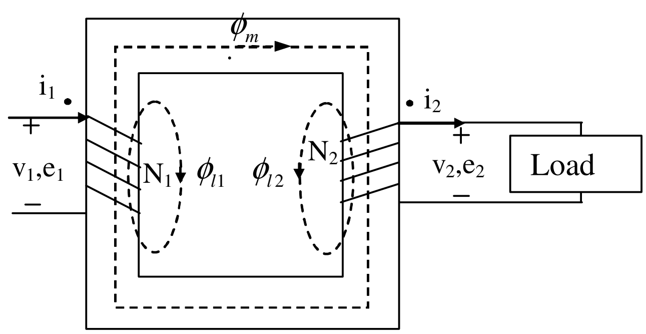


Figure 4.3 A practical transformer

The core of a practical transformer has finite permeability and core loss, so the primary winding draws the excitation current from the source even though there is no load attached to the secondary winding. The excitation current $\hat{I}_\phi$ is the sum of the core-loss current $\hat{I}_c$ and the magnetization current $\hat{I}_m$, that is,

$$\hat{I}_\phi = \hat{I}_c + \hat{I}_m$$

The core loss can be modelled by an equivalent core-loss resistance R_c and the magnetization effect can be described by an equivalent magnetizing reactance X_m. If the induced voltage across the primary winding is $\hat{E}_1$, then

$$\hat{I}_c = \frac{\hat{E}_1}{R_c}$$

$\widehat{I}_m = \frac{\widehat{E}_1}{jX_m}$

Note that the effective mutual flux created by $\widehat{I}_\phi$ should be equal to the mutual flux in the core. Assume that the reluctance of the core is $\Re$. Then, we have

$$\phi = \frac{N_1 \widehat{I}_\phi}{\Re} = \frac{N_1 \widehat{I}_1 - N_2 \widehat{I}_2}{\Re}$$

that is,

$$N_1 \widehat{I}_\phi = N_1 \widehat{I}_1 - N_2 \widehat{I}_2$$

Therefore, one gets

$$\widehat{I}_2' = \widehat{I}_1 - \widehat{I}_\phi = \frac{N_2}{N_1} \widehat{I}_2$$

which implies that the relationship among the quantities $\widehat{E}_1, \widehat{E}_2, \widehat{I}_2',$ and $\widehat{I}_2$ can be modelled by an ideal transformer, where $\widehat{I}_2'$ is the load current viewed from the primary side. The equivalent circuit for a practical transformer is shown in Figure 4.4.

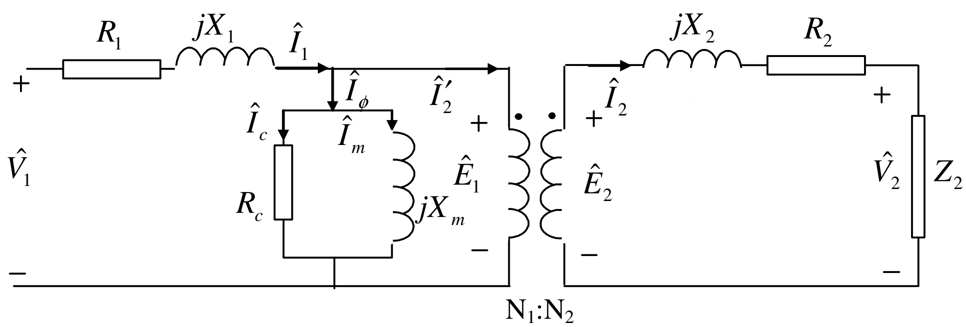


Figure 4.4 The equivalent circuit of a practical transformer

After the secondary is transformed to the primary side, the equivalent circuit becomes one as shown in Figure 4.5.

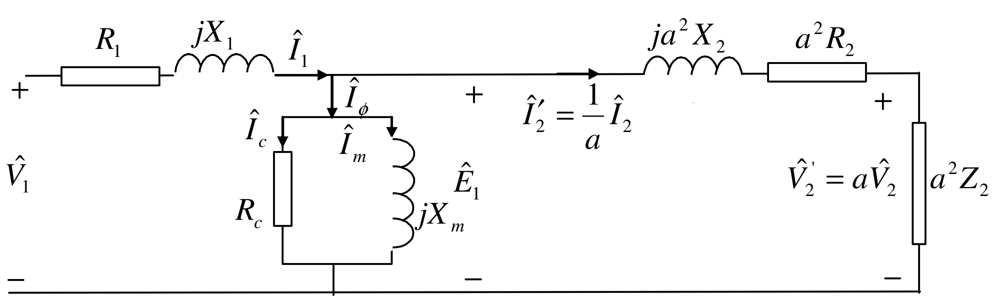


Figure 4.5 The equivalent circuit as viewed from the primary side

On the other hand, Figure 4.6 shows the equivalent circuit as viewed from the secondary side.

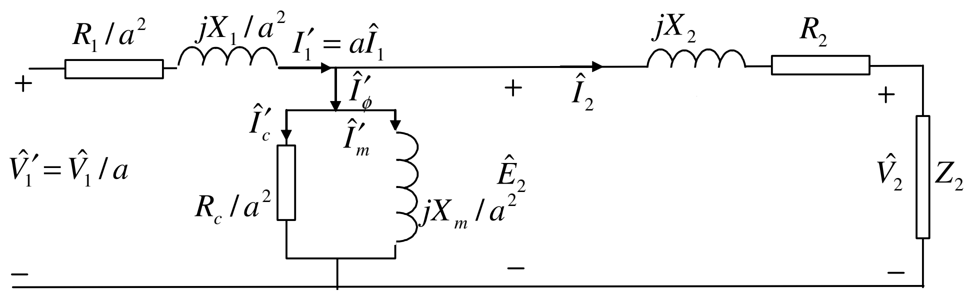


Figure 4.6 The equivalent circuit as viewed from the secondary side

In a well-designed transformer, $R_1, R_2, X_1,$ and X_2 are kept as small as possible, and R_c and X_m are kept as big as possible so that the transformer efficiency can be made as high as possible. Since R_1 and X_1 are quite low, the voltage drop across them is also low in comparison with the applied voltage. Without introducing any appreciable error, we can assume that the voltage across the parallel branch is the same as the applied voltage. This assumption allows us to move the parallel branch as shown in Figure 4.7.

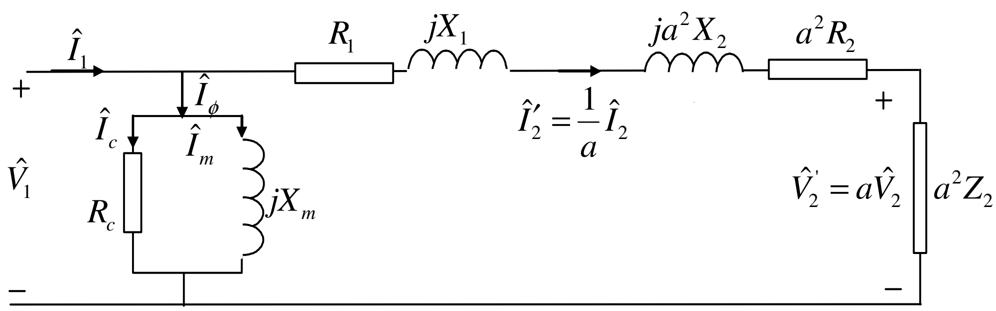


Figure 4.7 The approximate equivalent circuit as viewed from the primary

4.3 Voltage Regulation and Maximum Efficiency Criterion

The voltage regulation $VR\%$ is defined as

$$VR\% = \frac{V_{2NL} - V_{2FL}}{V_{2FL}} \times 100$$

where V_{2NL} and V_{2FL} are effective values of no-load and full-load voltages at the secondary terminals. For an ideal transformer, the voltage regulation is zero. The smaller the voltage regulation, the better the transformer.

The input power to an transformer is calculated by

$$S_{in} = \hat{V}_1 \hat{I}_1^*$$
$$P_{in} = V_1 I_1 \cos\theta_1$$
$$Q_{in} = V_1 I_1 \sin\theta_1$$

where $cos\theta_1$ is the power factor of the transformer and θ_1 is the power factor angle of the transformer which is the difference between the voltage phase angle and current phase angle.

It follows from the approximate equivalent circuit shown in Figure 4.7 that the output power is

$$P_o = I_2' V_2' \cos\theta_2$$

where $cos\theta_2$ is the power factor of the load and θ_2 is the power factor angle of the load.

The copper loss is

$$P_{cu} = (I'_2)^2(R_1 + a^2R_2)$$

Recall that the core loss is determined by $P_m = K_e f^2 B^2 + K_h f B^n$. The flux in the transformer is almost constant, so is B. Therefore, P_m is essentially constant. The input power can also determined by

$$P_{in} = P_o + P_{cu} + P_m = I'_2 V'_2 \cos\theta_2 + (I'_2)^2(R_1 + a^2R_2) + P_m$$

The efficiency of the transformer is

$$\eta = \frac{P_o}{P_{in}} = \frac{I'_2 V'_2 \cos\theta_2}{I'_2 V'_2 \cos\theta_2 + (I'_2)^2 (R_1 + a^2 R_2) + P_m}$$

which is a function of I'_2. To get the load current $I'_{2\eta}$ for the maximum efficiency, we differentiate η with respect with I'_2 and set it to be zero, that is,

$$\frac{d\eta}{dI_p} = \frac{V'_2\cos\theta_2\left(I'_2V'_2\cos\theta_2 + (I'_2)^2(R_1+a^2R_2)+P_m\right) - I'_2V'_2\cos\theta_2\left(V'_2\cos\theta_2 + 2I'_2(R_1+a^2R_2)\right)}{\left(I'_2V'_2\cos\theta_2+(I'_2)^2(R_1+a^2R_2)+P_m\right)^2} = \frac{V'_2\cos\theta_2\left(P_m - (I'_2)^2(R_1+a^2R_2)\right)}{\left(I'_2V'_2\cos\theta_2+(I'_2)^2(R_1+a^2R_2)+P_m\right)^2} =$$

which implies that

$$P_m = (I'_2)^2(R_1 + a^2R_2) = P_{cu}$$

The above equation indicates that the efficiency of a transformer is maximum when the copper loss is equal to the core loss. The load current $I'_{2\eta}$ for the maximum efficiency is given by

$$I'_{2\eta} = \sqrt{\frac{P_m}{R_1 + a^2R_2}}$$

4.4 Determination of Transformer Parameters

Suppose a step-down transformer is tested in this section.

The Short-Circuit Test

Short-circuit the low-voltage side, increase the voltage on the high-voltage side until the rated current is reached on the low-voltage side, and measure the voltage, current, and power on the high-voltage side. The equivalent circuit for the short-circuit test is shown in Figure 4.8.

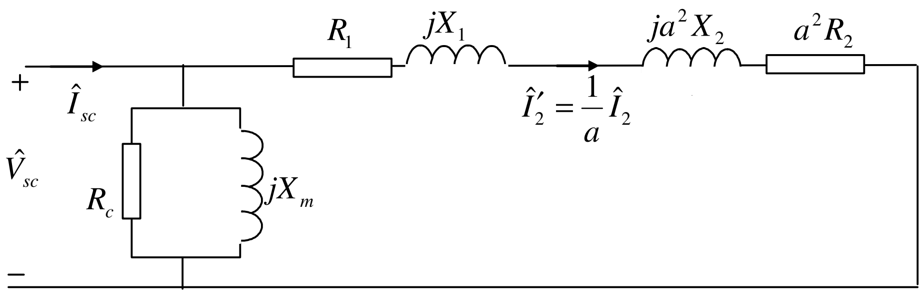


Figure 4.8 The equivalent circuit for the short-circuit test

Define $Z_{eq} = R_{eq} + jX_{eq} = R_1 + a^2R_2 + j(X_1 + a^2X_2)$. Then, it follows from the equivalent circuit that

$$R_{eq} = R_1 + a^2R_2 = \frac{P_{sc}}{I_{sc}^2}$$

$$|Z_{eq}| = \frac{V_{sc}}{I_{sc}}$$

$$X_{eq} = X_1 + a^2X_2 = \sqrt{|Z_{eq}|^2 - R_{eq}^2}$$

For most transformers, resistances and reactances can be separated by

$$R_1 = a^2R_2 = 0.5R_{eq}$$

$X_1 = a^2X_2 = 0.5X_{eq}$

The Open-Circuit Test

Open-circuit the high-voltage side, apply the rated voltage to the low-voltage side, and measure the voltage, current, and power on the low-voltage side. The equivalent circuit for the open-circuit test is shown in Figure 4.9.

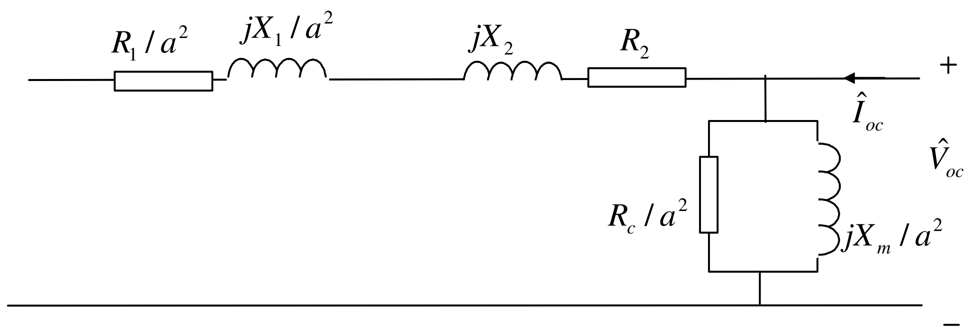


Figure 4.9 The equivalent circuit for the open-circuit test

Let $Z_\phi = \frac{1}{\frac{1}{R_c/a^2}+\frac{1}{jX_m/a^2}}$. Then it follows from the equivalent circuit that

$$R_c/a^2 = \frac{V_{oc}^2}{P_{oc}} \Rightarrow R_c = a^2\frac{V_{oc}^2}{P_{oc}}$$

$$|Z_\phi| = \frac{V_{oc}}{I_{oc}}$$

Note that $\frac{1}{Z_\phi} = \frac{1}{R_c/a^2} + \frac{1}{jX_m/a^2}$, i.e. $\left|\frac{1}{Z_\phi}\right|^2 = \left(\frac{1}{R_c/a^2}\right)^2 + \left(\frac{1}{X_m/a^2}\right)^2$. As a result, we have

$$X_m/a^2 = \frac{1}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2 - \left(\frac{1}{R_c/a^2}\right)^2}} \Rightarrow X_m = \frac{a^2}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2 - \left(\frac{1}{R_c/a^2}\right)^2}}$$

Example 4.1 A 50kVA 2400:240V transformer is tested and the following data were recorded: the short-circuit test readings with the low-voltage side short-circuited are 48V, 20.8A, and 617W; the open-circuit test readings with the high-voltage side open-circuited are 240V, 5.41A, and 186W. Find the transformer parameters, the efficiency, and the voltage regulation at full load and 0.8 power factor lagging. Determine the load current for the maximum efficiency.

Solution: The transformation ratio is $a = \frac{V_1}{V_2} = \frac{2400}{240} = 10$. The approximate equivalent circuit is shown in Figure 4.7. From the short-circuit test (see Figure 4.8),

$V_{sc} = 48V, I_{sc} = 20.8A, P_{sc} = 617W$

$$R_{eq} = \frac{P_{sc}}{I_{sc}^2} = \frac{617}{20.8^2} = 1.4261\Omega$$

$$|Z_{eq}| = \frac{V_{sc}}{I_{sc}} = \frac{48}{20.8} = 2.3077\Omega$$

$$X_{eq} = \sqrt{|Z_{eq}|^2 - R_{eq}^2} = \sqrt{2.3077^2 - 1.4261^2} = 1.8143\Omega$$

Therefore,

$$R_1 = 0.5R_{eq} = 0.5 \times 1.4261 = 0.71305\Omega$$

$$a^2R_2 = 0.5R_{eq} = 0.5 \times 1.4261 = 0.71305\Omega$$

$$X_1 = 0.5X_{eq} = 0.5 \times 1.8143 = 0.90715\Omega$$

$$a^2X_2 = 0.5X_{eq} = 0.5 \times 1.8143 = 0.90715\Omega$$

For the open-circuit test, the equivalent circuit is shown in Figure 4.9. From the open-circuit test, $V_{oc} = 240V$, $I_{oc} = 5.41A$, $P_{oc} = 186W$. Therefore,

$$R_c = a^2\frac{V_{oc}^2}{P_{oc}} = 10^2 \times \frac{240^2}{186} = 30968\Omega$$

$|Z_\phi| = \frac{V_{oc}}{I_{oc}} = \frac{240}{5.41} = 44.362\Omega$

$X_m = \frac{a^2}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2 - \left(\frac{1}{R_C/a^2}\right)^2}} = \frac{10^2}{\sqrt{\left|\frac{1}{44.362}\right|^2 - \left(\frac{1}{30968/10^2}\right)^2}} = 4482.4\Omega$

The full load current is

$I_2 = \frac{S_2}{V_2} = \frac{50000}{240} = 208A$

The load current referred to the primary side is

$I_2' = \frac{1}{a}I_2 = \frac{1}{10}208.33 = 20.8A$

The core loss is the same as the input power in the open-circuit test, that is,

$P_m = 168W$

The copper loss is

$P_{cu} = (I_2')^2(R_1 + a^2R_2) = 20.8^2 \times 1.4261 = 617W$

The output power at full load is

$P_o = V_2I_2\cos\theta_2 = 240 \times 208 \times 0.8 = 39936W$

The input power at full load is

$P_{in} = P_o + P_{cu} + P_m = 39936 + 617 + 168 = 40721W$

The efficiency at full load is

$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{39936}{40721} \times 100 = 98.0\%$

The load voltage phasor is chosen as a reference, that is, $\widehat{V}_2 = 240\angle 0^\circ V$ and $\widehat{V}_2' = a\widehat{V}_2 = 2400\angle 0^\circ V$. Then, the load current phasor is

$\widehat{I}_2 = I_2\angle\cos^{-1}(0.8) = 208\angle -36.9^\circ A$

The load current phasor referred to the primary side is

$\widehat{I}_2' = \frac{1}{a}\widehat{I}_2 = \frac{1}{10}208\angle -36.9^\circ = 20.8\angle -36.9^\circ = 20.8(0.8 - j0.6)A$

It follows from Figure 4.6 that

$\widehat{V}_1 = \widehat{V}_2' + \widehat{I}_2'(R_1 + a^2R_2 + j(X_1 + a^2X_2))$

$= 2400 + 20.8(0.8 - j0.6)(1.4261 + j1.8143) = 2446.3 + j12.392$

$= \sqrt{12.392^2 + 2446.3^2}\angle\frac{180}{\pi}\tan^{-1}\left(\frac{12.392}{2446.3}\right) = 2446.3\angle\ 0.29^\circ V$

Now let us find the no load output voltage corresponding to $\widehat{V}_1 = 2446.3\angle\ 0.29^\circ V$ by using the approximate equivalent circuit. It is obvious that
$V_2' = V_1 = 2446.3$ and $V_2 = V_2'/a = 244.63V$. Therefore, the voltage regulation is

$VR\% = \frac{244.63 - 240}{240} \times 100 = 1.93\%$

The load current for the maximum efficiency viewed from the primary side is

$I_{2\eta}' = \sqrt{\frac{P_m}{R_1 + a^2R_2}} = \sqrt{\frac{168}{1.4261}} = 10.854A$

and the load current for the maximum efficiency is

$I_{2\eta} = aI_{2\eta}' = 108.54A$

4.5 Per-Unit Computations

quantities such as voltage, current, power, reactive power, voltamperes, resistance, reactance, and impedance can be translated to and from per-unit form as follows:

$$\text{Quantity in per-unit} = \frac{\text{Actual quantity}}{\text{Base Value of quantity}}$$

For a single phase system, the base values must obey the electric circuit laws, that is,

$P_{base}, Q_{base}, VA_{base} = V_{base}I_{base}$

$R_{base}, X_{base}, Z_{base} = \frac{V_{base}}{I_{base}}$

1. Select a VA base and a base voltage at some point in the system.
2. Convert all quantities to per-unit.
3. Perform a standard electrical analysis with all quantities in per-unit.
4. Convert all quantities back to real units by multiplying their per-unit values by their corresponsing base values.

Note that the turns ratio of an ideal transformer in per unit is one.

Example 4.2: A single-phase generator with an internal impedance $Z_g = 23 + j92m\Omega$ is connected to a load via a 46kVA, 230/2300V, step-up tansformer, a short transmission line and a 46kVA, 2300/115V, sstep-down transformer. The impedance of the transmission line is $Z_{tl} = 2.07 + j4.14\Omega$. The parameters of step-up and step-down transformers are:

$Z_{1g} = 23 + j69m\Omega$, $Z_{\phi g} = 138 + j69\Omega$, $Z_{2g} = 2.3 + j6.9\Omega$, $Z_{1l} = 2.33 + j6.9\Omega$,
$Z_{\phi l} = 11.5 + j9.2k\Omega$, $Z_{2l} = 5.75 + j17.25m\Omega$.

Determine (a) the generator voltage, (b) the generator current, and (c) the overall efficiency of the system at full load and 0.866 pf lagging.

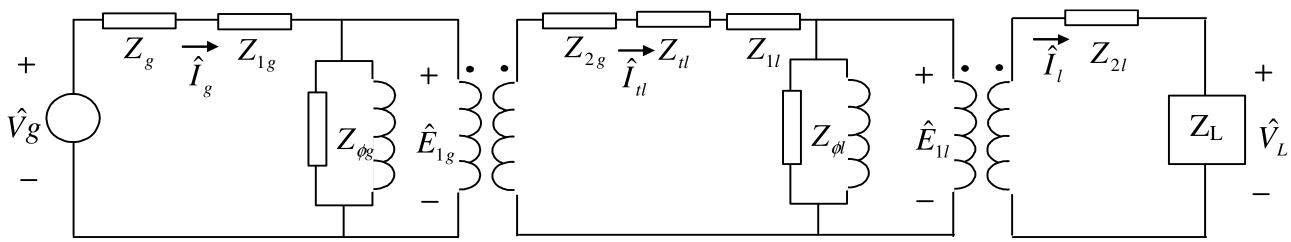


Figure 4.10 The circuit for Example 4.2

Solution:The equivalent circuit of the system incorperating ideal transformers is given in Figure 4.10.

For the generator side, choose the base values $V_{bg} = 230V$ and $S_{bg} = 46000VA$. Then, we have

$$I_{bg} = \frac{S_{bg}}{V_{bg}} = \frac{46000}{230} = 200A$$

$$Z_{bg} = \frac{V_{bg}}{I_{bg}} = \frac{230}{200} = 1.15\Omega$$

The per-unit impedance of the generator is

$$Z_{g,pu} = \frac{Z_g}{Z_{bg}} = \frac{0.023+j0.092}{1.15} = 0.02 + j0.08$$

The per-unit parameters on the primary side of the step-up transformer are

$$Z_{1g,pu} = \frac{Z_{1g}}{Z_{bg}} = \frac{0.023+j0.069}{1.15} = 0.02 + j0.06$$

$$Z_{\phi g,pu} = \frac{Z_{\phi g}}{Z_{bg}} = \frac{138+j69}{1.15} = 120 + j60$$

For the transmission line side, choose the base values $V_{btl} = 2300V$ and $S_{btl} = 46000VA$. Then, we have

$$I_{btl} = \frac{S_{btl}}{V_{btl}} = \frac{46000}{2300} = 20A$$

$$Z_{btl} = \frac{V_{btl}}{I_{btl}} = \frac{2300}{20} = 115\Omega$$

The per-unit impedance on the secondary side of the step-up transformer is

$$Z_{2g,pu} = \frac{Z_{2g}}{Z_{btl}} = \frac{2.3+j6.9}{115} = 0.02 + j0.06$$

The per-unit impedance of the transmission line is

$$Z_{tl,pu} = \frac{Z_{tl}}{Z_{btl}} = \frac{2.07+j4.14}{115} = 0.018 + j0.036$$

The per-unit parameters on the primary side of the step-down transformer are

$$Z_{1l,pu} = \frac{Z_{1l}}{Z_{btl}} = \frac{2.3+j6.9}{115} = 0.02 + j0.06$$

$$Z_{\phi l,pu} = \frac{Z_{\phi l}}{Z_{btl}} = \frac{11500+j9200}{115} = 100 + j80$$

For the load side, choose the base values $V_{bl} = 115V$ and $S_{bl} = 46000VA$. Then, we have

$$I_{bl} = \frac{S_{bl}}{V_{bl}} = \frac{46000}{115} = 400A$$

$$Z_{bl} = \frac{V_{bl}}{I_{bl}} = \frac{115}{400} = 0.2875\Omega$$

The per-unit impedance on the secondary side of the step-down transformer is

$$Z_{2l,pu} = \frac{Z_{2l}}{Z_{bl}} = \frac{0.00575+j0.01725}{0.2875} = 0.02 + j0.06$$

The per-unit load voltage and per-unit load current are

$$V_{l,pu} = \frac{V_l}{V_{bl}} = \frac{115}{115} = 1$$

$$I_{l,pu} = \frac{I_l}{I_{bl}} = \frac{\frac{S_l}{V_l}}{400} = \frac{\frac{46000}{115}}{400} = 1$$

The load voltage and current phasors are

$$\hat{V}_{l,pu} = 1\angle 0^\circ$$

$$\hat{I}_{l,pu} = 1\angle -30^\circ$$

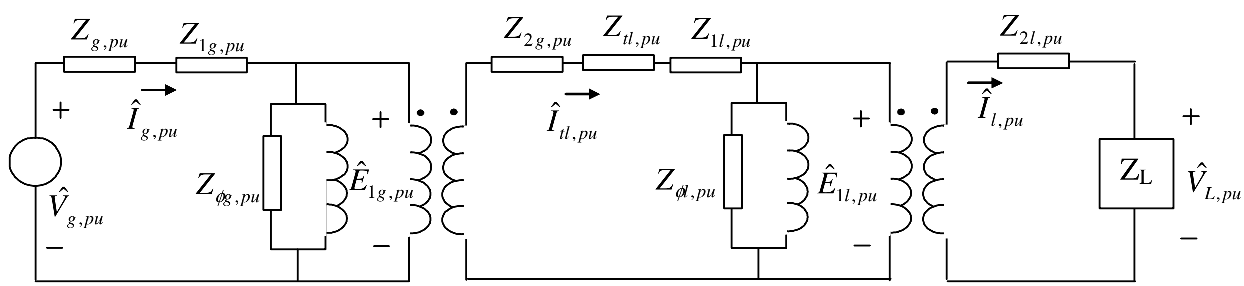


Figure 4.11 The equivalent circuit in per-unit for Example 4.2

The equivalent circuit of the system in per unit is shown in Figure 4.11. It follows from this equivalent circuit that

$$\hat{E}_{1l,pu} = \hat{I}_{l,pu} Z_{2l,pu} + \hat{V}_{l,pu} = (1\angle -30^\circ)(0.02 + j0.06) + 1\angle 0^\circ$$

$$= \left(\cos\left(\tfrac{\pi}{6}\right) - j\sin\left(\tfrac{\pi}{6}\right)\right)(0.02 + j0.06) + 1 = 1.047 + j0.042$$

$$\hat{I}_{tl,pu} = \hat{I}_{l,pu} + \frac{\hat{E}_{1l,pu}}{Z_{\phi l,pu}} = 1\angle -30^\circ + \frac{1.047+j0.042}{100+j80} = \cos\left(\tfrac{\pi}{6}\right) - j\sin\left(\tfrac{\pi}{6}\right) + \frac{(1.047+j0.042)(100-j80)}{100^2+80^2} =$$

$0.872 - j0.505$

$$\hat{E}_{1g,pu} = \hat{I}_{tl,pu}(Z_{2g,pu} + Z_{tl,pu} + Z_{1l,pu}) + \hat{E}_{1l,pu}$$

$$= (0.872 - j0.505)(0.02 + j0.06 + 0.018 + j0.036 + 0.02 + j0.06) + 1.047 + j0.042$$

$$= 1.176 + j0.149$$

$$\hat{I}_{g,pu} = \hat{I}_{tl,pu} + \frac{\hat{E}_{1g,pu}}{Z_{\phi g,pu}} = 0.872 - j0.505 + \frac{1.176+j0.149}{120+j60} = 0.872 - j0.505 + \frac{(1.176+j0.149)(120-j60)}{120^2+60^2}$$

$$= 0.880 - j0.508$$

$$\hat{V}_{g,pu} = \hat{I}_{g,pu}(Z_{g,pu} + Z_{1g,pu}) + \hat{E}_{1g,pu} = (0.880 - j0.508)(0.02 + j0.08 + 0.02 + j0.06) + 1.$$

$176 + j0.149$

$$= 1.282 + j0.252$$

(a) Therefore, the generator voltage is

$\hat{V}_g = V_{bg}\hat{V}_{g,pu} = 230(1.282 + j0.252) = 294.86 + j57.$
$96 = \sqrt{294.86^2 + 57.96^2} \angle \tan^{-1}\left(\frac{57.96}{294.86}\right)\frac{180}{\pi}$

$= 300.5\angle 11.1^\circ V$

(b) The generator current is

$\hat{I}_g = I_{bg}\hat{I}_{g,pu} = 200(0.880 - j0.508) = 176.0 - j101.$
$6 = \sqrt{176.0^2 + 101.6^2} \angle \tan^{-1}\left(\frac{-101.6}{176.0}\right)\frac{180}{\pi}$

$= 203.2\angle -30^\circ A$

(c) On a per-unit basis, the rated power output at a 0.866 pf lagging is

$$P_{o,pu} = \hat{V}_{l,pu}\hat{I}_{l,pu}\cos\theta = 0.866$$

The per-unit input power from the generator is

$$P_{in,pu} = \mathrm{Re}\left[\hat{V}_{g,pu}\hat{I}^*_{g,pu}\right] = \mathrm{Re}[(1.282 + j0.252)(0.880 + j0.508)] = 1.0002$$

Thus, the efficiency is

$$\eta = \frac{P_{o,pu}}{P_{in,pu}} \times 100 = \frac{0.866}{1.0002} \times 100 = 86.6\%$$

4.6 Autotransformers

An ideal two-winding transformer can be connected as an ideal autotransformer. There are four possible ways to connect a two-winding transformer as an autotransformer, as shown in Figures 4.12-4.15.

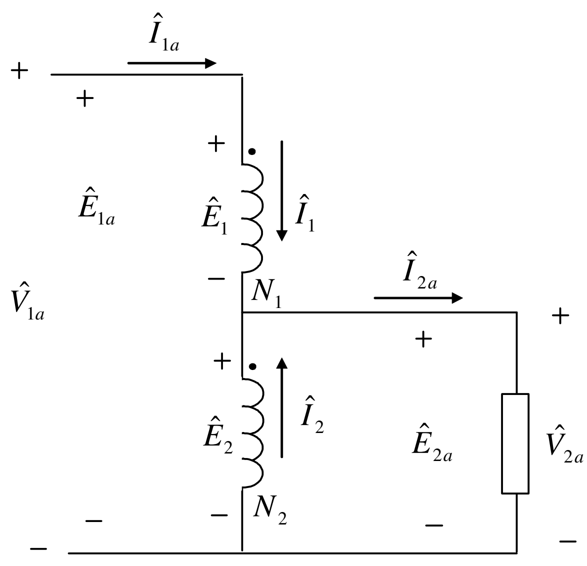


Figure 4.12 A step-down authotransformer

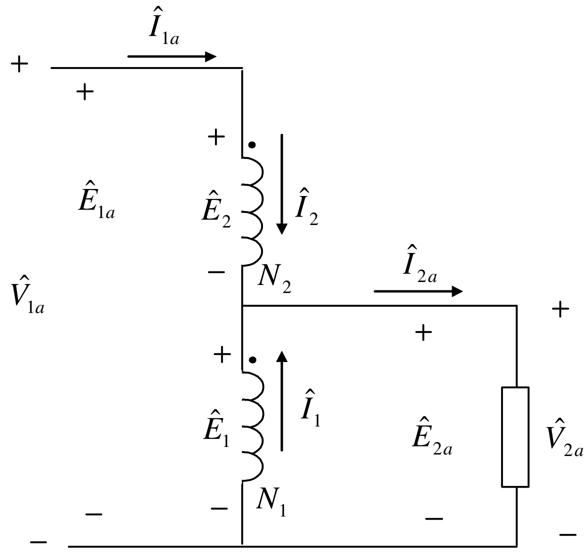


Figure 4.13 A step-down authotransformer

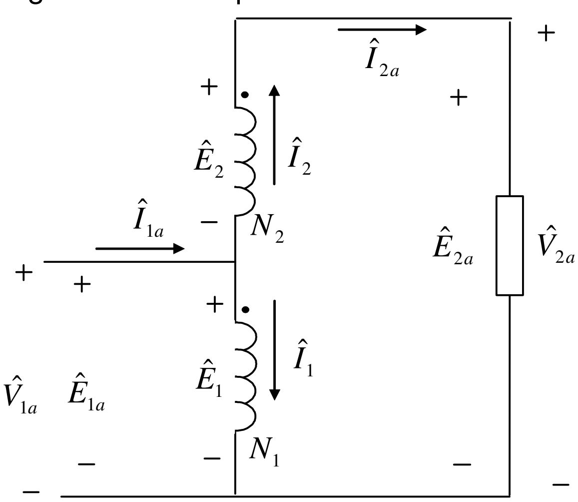


Figure 4.14 A step-up autotransformer

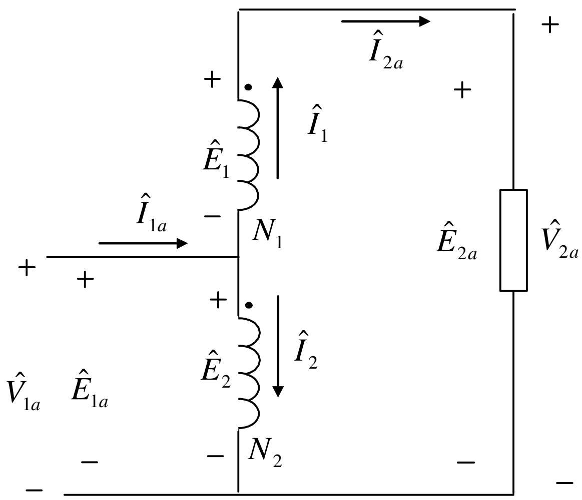


Figure 4.15 A step-up autotransformer

The following examples show how to calculate the primary winding voltage and current, $\hat{V}_{1a}$ and $\hat{I}_{1a}$, the secondary winding voltage and current, $\hat{V}_{2a}$ and $\hat{I}_{2a}$, the ratio of transformation a_T, and the apparent power input and output, $\hat{S}_{ina}$ and $\hat{S}_{oa}$.

Example 4.3: A 24kVA 2400/240V two-winding transformer is to be connected as an autotransformer. For each possible combination, determine the primary winding voltage and current, V_{1a} and I_{1a}, the secondary winding voltage and current, V_{2a} and I_{2a}, the ratio of transformation a_T, and the apparent power input and output, S_{ina} and S_{oa} under ideal conditions.

Solution: For the given information for the two-winding transformer, we get

$$E_1 = V_1 = 2400V, E_2 = V_2 = 240V, a = \frac{V_1}{V_2} = 10, S_o = 24000VA, I_2 = \frac{S_o}{V_2} = 100A, I_1 = \frac{I_2}{a} = 10A$$

For the autotransformer shown in Figure 4.12,

$$V_{1a} = E_{1a} = E_1 + E_2 = 2640V, V_{2a} = E_{2a} = E_2 = 240V, a_T = \frac{V_{1a}}{V_{2a}} = 11, I_{1a} = I_1 = 10A, I_{2a} = I_1 + I$$

$$S_{ina} = V_{1a}I_{1a} = 2640 \times 10 = 26400VA, S_{oa} = V_{2a}I_{2a} = 240 \times 110 = 26400VA$$

The nominal rating of the autotransformer in Figure 4.12 is 26.4kVA, 2640/240V.

For the autotransformer shown in Figure 4.13,

$$V_{1a} = E_{1a} = E_2 = 240V, V_{2a} = E_{2a} = E_1 + E_2 = 2640V, a_T = \frac{V_{1a}}{V_{2a}} = 0.091, I_{1a} = I_1 + I_2 = 110A, I_{2a}$$

$$S_{ina} = V_{1a}I_{1a} = 240 \times 110 = 26400VA, S_{oa} = V_{2a}I_{2a} = 2640 \times 10 = 26400VA$$

The nominal rating of the autotransformer in Figure 4.13 is 26.4kVA, 240/2640V.

For the autotransformer shown in Figure 4.14,

$$V_{1a} = E_{1a} = E_1 + E_2 = 2640V, V_{2a} = E_{2a} = E_1 = 2400V, a_T = \frac{V_{1a}}{V_{2a}} = \frac{2640}{2400} = 1.1, I_{1a} = I_2 = 100A,$$

$$S_{ina} = V_{1a}I_{1a} = 2640 \times 100 = 264000VA, S_{oa} = V_{2a}I_{2a} = 2400 \times 110 = 264000VA$$

The nominal rating of the autotransformer in Figure 4.14 is 264kVA, 2640/2400V.

For the autotransformer shown in Figure 4.15,

$$V_{1a} = E_{1a} = E_1 = 2400V, V_{2a} = E_{2a} = E_1 + E_2 = 2640V, a_T = \frac{V_{1a}}{V_{2a}} = \frac{2400}{2640} = 0.91, I_{1a} = I_1 + I_2 = 110A$$

$$S_{ina} = V_{1a}I_{1a} = 2400 \times 110 = 264000VA, S_{oa} = V_{2a}I_{2a} = 2640 \times 100 = 264000VA$$

The nominal rating of the autotransformer in Figure 4.15 is 264kVA, 2400/2640V.

Note that the nominal rating of the autotransformer in Figure 4.14 or 4.15 is 10 times the nominal rating of the two-winding transformer.

Example 4.4: A 720VA 360/120V two-winding transformer has the following parameters: $R_1 = 18.9\Omega$, $X_1 = 21.6\Omega$, $R_2 = 2.1\Omega$, $X_2 = 2.4\Omega$, $R_{c1} = 8.64k\Omega$, $X_{m1} = 6.84k\Omega$. The transformer is connected as a 120/480V step-up autotransformer. If the autotransformer delivers the full load at 0.707 pf leading, determine its efficiency and voltage regulation.

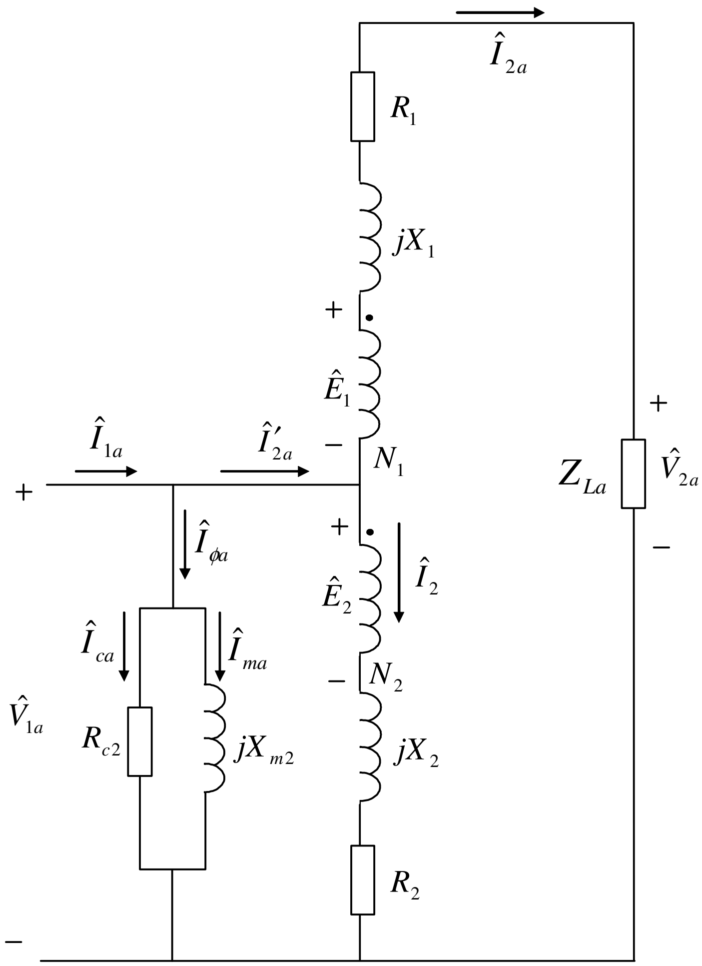


Figure 4.16 The approximate equivalent circuit of a

Solution: The turns ration of the two-winding transformer is $a = \frac{360}{120} = 3$ and the turns ratio of the autotransformer is $a_T = \frac{120}{480} = 0.25$. The equivalent core-loss resistance and the magnetizing reactance on the secondary side of the two-winding transformer is

$$R_{c2} = \frac{R_{c1}}{a^2} = \frac{8640}{3^2} = 960\Omega,\ X_{m2} = \frac{X_{m1}}{a^2} = \frac{6840}{3^2} = 760\Omega$$

The approximate equivalent circuit of the autotransformer is shown in Figure 4.16.

Assume that $\widehat{V}_{2a} = 480\angle 0^\circ V$. The full load current is

$$I_{2a} = I_1 = \frac{S_{in}}{V_1} = \frac{720}{360} = 2A, \text{ so } \widehat{I}_{2a} = 2\angle 45^\circ A = 2\cos\left(45\frac{\pi}{180}\right) + j2\sin\left(45\frac{\pi}{180}\right)$$

and $\widehat{I}'_{2a} = \frac{\widehat{I}_{2a}}{a_T} = \frac{2}{0.25} = 8\angle 45^\circ A = 8\cos\left(45\frac{\pi}{180}\right) + j8\sin\left(45\frac{\pi}{180}\right)$.

Hence, $\widehat{I}_2 = \widehat{I}'_{2a} - \widehat{I}_{2a} = 8\angle 45^\circ - 2\angle 45^\circ =$

$$8\cos\left(45\frac{\pi}{180}\right) + j8\sin\left(45\frac{\pi}{180}\right) - 2\cos\left(45\frac{\pi}{180}\right) - j2\sin\left(45\frac{\pi}{180}\right) = 6\cos\left(45\frac{\pi}{180}\right) + j6\sin\left(45\frac{\pi}{180}\right)$$

Note that $\widehat{E}_1 = a\widehat{E}_2 = 3\widehat{E}_2$. Then, it follows from KVL that

$$\widehat{E}_1 - \widehat{I}_{2a}(R_1 + jX_1) - \widehat{V}_{2a} + \widehat{I}_2(R_2 + jX_2) + \widehat{E}_2 = 0$$

or

$$\widehat{E}_2 = \tfrac{1}{4}\left(\widehat{I}_{2a}(R_1 + jX_1) + \widehat{V}_{2a} - \widehat{I}_2(R_2 + jX_2)\right)$$
$$= \tfrac{1}{4}\Big(\Big(2\cos\Big(45\tfrac{\pi}{180}\Big) + j2\sin\Big(45\tfrac{\pi}{180}\Big)\Big)(18.9 + j21.6) + 480 - \Big(6\cos\Big(45\tfrac{\pi}{180}\Big) + j6\sin\Big(45\tfrac{\pi}{180}$$
$$= 119.36 + j9.5459V$$

Thus,

$$\widehat{V}_{1a} = \widehat{E}_2 + \widehat{I}_2(R_2 + jX_2) = 119.36 + j9.5459 + \Big(6\cos\Big(45\tfrac{\pi}{180}\Big) + j6\sin\Big(45\tfrac{\pi}{180}\Big)\Big)(2.1 + j2.4)$$
$$= 118.09 + j28.638 = \sqrt{118.09^2 + 28.638^2}\angle\tan^{-1}\Big(\tfrac{28.638}{118.09}\Big)\tfrac{180}{\pi} = 121.51\angle 13.6^\circ V$$

On the other hand,

$$\widehat{I}_{1a} = \widehat{I}_{\phi a} + \widehat{I}'_{2a} = \tfrac{\widehat{V}_{1a}}{R_{c2}} + \tfrac{\widehat{V}_{1a}}{jX_{c2}} + \widehat{I}'_{2a} = \widehat{V}_{1a}\Big(\tfrac{1}{R_{c2}} + \tfrac{1}{jX_{c2}}\Big) + \widehat{I}'_{2a}$$
$$= (118.09 + j28.638)\Big(\tfrac{1}{960} + \tfrac{1}{j760}\Big) + 8\cos\Big(45\tfrac{\pi}{180}\Big) + j8\sin\Big(45\tfrac{\pi}{180}\Big)$$
$$= (118.09 + j28.638)\Big(\tfrac{1}{960} - j\tfrac{1}{760}\Big) + 8\cos\Big(45\tfrac{\pi}{180}\Big) + j8\sin\Big(45\tfrac{\pi}{180}\Big)$$
$$= 5.8176 + j5.5313$$

Therefore,

$$P_o = \mathrm{Re}\Big(\widehat{V}_{2a}\widehat{I}^*_{2a}\Big) = \mathrm{Re}\Big((480)\Big(2\cos\Big(45\tfrac{\pi}{180}\Big) - j2\sin\Big(45\tfrac{\pi}{180}\Big)\Big)\Big) = 678.82W$$
$$P_{in} = \mathrm{Re}\Big(\widehat{V}_{1a}\widehat{I}^*_{1a}\Big) = \mathrm{Re}((118.09 + j28.638)(5.8176 - j5.5313)) = 845.41W$$
$$\eta = \tfrac{P_o}{P_{in}} \times 100 = \tfrac{678.82}{845.41} \times 100 = 80.3\%$$

If we remove the load, the no-load voltage at the secondary of the autotransformer is

$$V_{2aNL} = \tfrac{V_{1a}}{a_T} = \tfrac{121.51}{0.25} = 486.04V$$

We now can compute the voltage regulation as

$$VR\% = \tfrac{V_{2aNL} - V_{2aFL}}{V_{2aFL}} \times 100 = \tfrac{486.04 - 480}{480} \times 100 = 1.24\%$$

4.7 Three-Phase Transformers

The three windings on either side of a three-phase transformer can be connected either in Y or in $\triangle$. Therefore, a three-phase transformer can be connected in four possible ways: Y/Y, $\triangle/\triangle$, $\triangle/Y$, $Y/\triangle$, as shown in Figures 17-20.

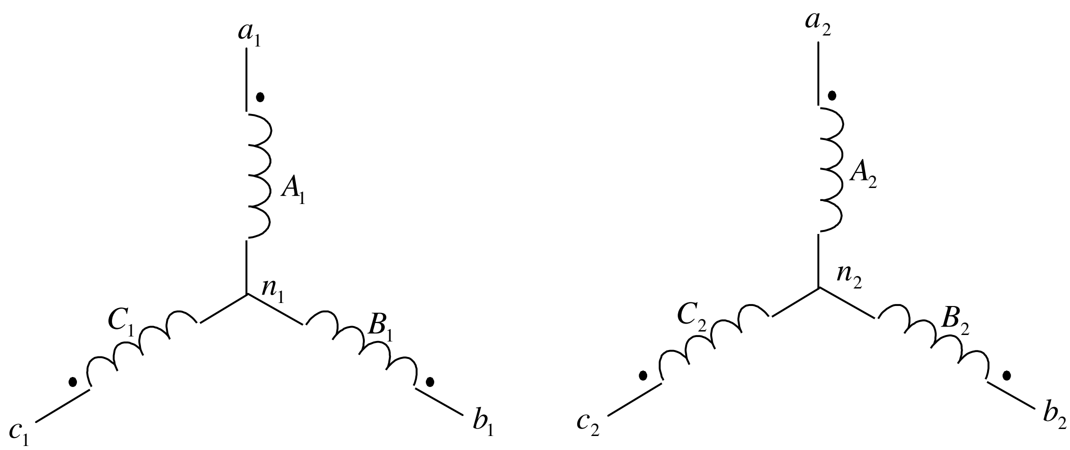


Figure 4.17 Y-Y connection

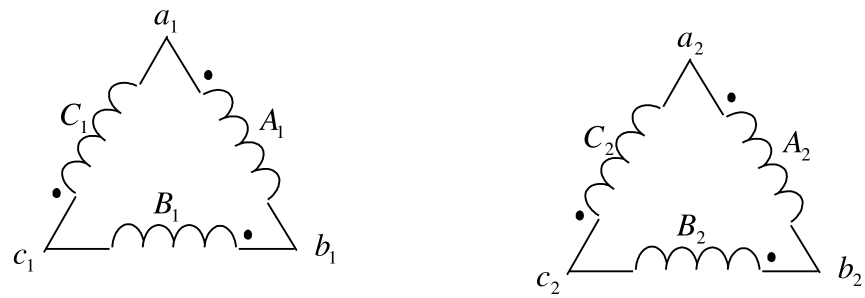


Figure 4.18 $\triangle - \triangle$ connection

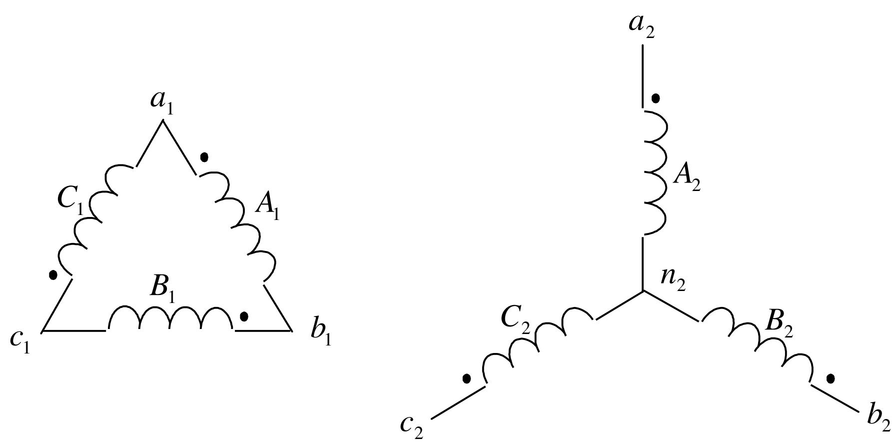


Figure 4.19 $\triangle - Y$ connection

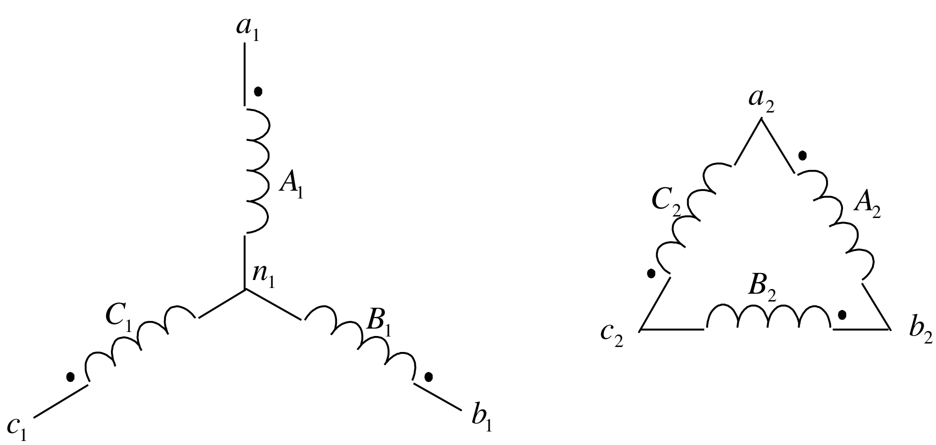


Figure 4.20 Y-$\triangle$connection

For simplicity, we assume that the three-phase transformer delivers a balanced load. Under steady-state conditions, a three-phase transformer can be analyzed by (a) transforming a $\triangle$-connected winding to a Y-connected winding using $\triangle$-to-Y transformation, (b) drawing a per-phase equivalent circuit, and (c) computing quantities for the per-phase equivalent circuit.

$\triangle$-to-Y transformation: If $Z_{\triangle}$ is the impedance in a $\triangle$-connected winding, the equivalent impedance Z_Y in a Y-connected winding is

$$Z_Y = \frac{1}{3} Z_\triangle$$

and

$$\widehat{V}_{an} = \frac{\widehat{V}_{ab}}{\sqrt{3}} \angle -30^\circ$$

$$\widehat{I}_{A'} = \sqrt{3}\,\widehat{I}_A \angle -30^\circ$$

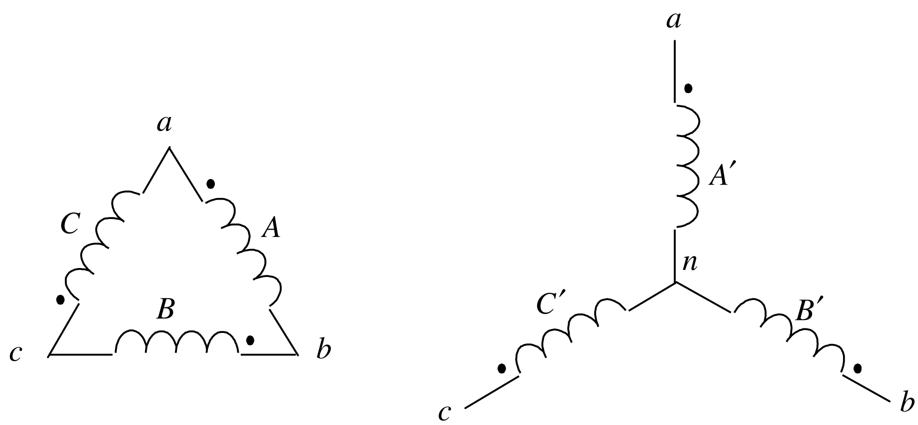
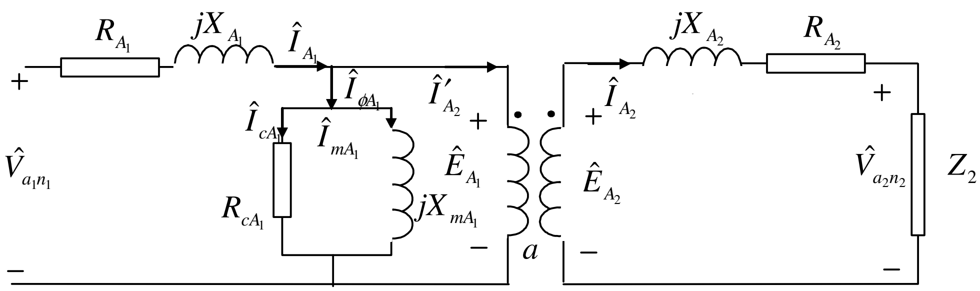


Figure 4.21 $\triangle - Y$ transformation

Example 4.5: A three-phase transformer is assembled by connecting three 720VA 360/120V single-phase transformers. The constants for each transformer are $R_H = 18.9\Omega$, $X_H = 21.6\Omega$, $R_L = 2.1\Omega$, $X_L = 2.4\Omega$, $R_{cH} = 8.64k\Omega$, $X_{mH} = 6.84k\Omega$. For each of the four configurations, determine the nominal voltage and power ratings of the three-phase transformer. Draw the per-phase equivalent circuit for each configuration.



Solution: The power rating of the three-phase transformer for each configuration is $S_{3\phi} = 3 \times 720 = 2160VA$.

(**a**) **Y**-**Y connection**: The nominal values of the line voltages on the primary and the secondary sides are

$$V_{1L} = \sqrt{3}\,V_{a_1n_1} = \sqrt{3} \times 360 = 623.54V$$

$$V_{2L} = \sqrt{3}\,V_{a_2n_2} = \sqrt{3} \times 120 = 207.85V$$

Thus, the nominal ratings of the three-phase transformer are 2.16kVA 624/208V Y/Y connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$a = \frac{360}{120} = 3$, $R_{A_1} = R_H = 18.9\Omega$, $X_{A_1} = X_H = 21.6\Omega$, $R_{A_2} = R_L = 2.1\Omega$, $X_{A_2} = X_L = 2.4\Omega$, $R_{cA_1} = R_{cH} = 8.64k\Omega$, $X_{mA_1} = X_{mH} = 6.84k\Omega$

(**b**) △-△ **connection**: The nominal values of the line voltages on the primary and the secondary sides are

$$V_{1L} = V_{a_1b_1} = 360V$$
$$V_{2L} = V_{a_2b_2} = 120V$$

Thus, the nominal ratings of the three-phase transformer are 2.16kVA 360/120V △/△ connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$$a = \frac{\frac{360}{\sqrt{3}}}{\frac{120}{\sqrt{3}}} = 3,\ R_{A_1} = \frac{R_H}{3} = \frac{18.9}{3} = 6.3\Omega,\ X_{A_1} = \frac{X_H}{3} = \frac{21.6}{3} = 7.2\Omega,\ R_{A_2} = \frac{R_L}{3} = \frac{2.1}{3} = 0.7\Omega,$$
$$X_{A_2} = \frac{X_L}{3} = \frac{2.4}{3} = 0.8\Omega,\ R_{cA_1} = \frac{R_{cH}}{3} = \frac{8.64}{3} = 2.88k\Omega,\ X_{mA_1} = \frac{X_{mH}}{3} = \frac{6.84}{3} = 2.28k\Omega$$

(**c**) △-Y **connection**: The nominal values of the line voltages on the primary and the secondary sides are

$$V_{1L} = V_{a_1b_1} = 360V$$
$$V_{2L} = \sqrt{3}\,V_{a_2n_2} = \sqrt{3} \times 120 = 207.85V$$

Thus, the nominal ratings of the three-phase transformer are 2.16kVA 360/208V △/Y connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$$a = \frac{\frac{360}{\sqrt{3}}}{120} = 1.732,\ R_{A_1} = \frac{R_H}{3} = \frac{18.9}{3} = 6.3\Omega,\ X_{A_1} = \frac{X_H}{3} = \frac{21.6}{3} = 7.2\Omega,\ R_{A_2} = R_L = 2.1\Omega,$$
$$X_{A_2} = X_L = 2.4\Omega,\ R_{cA_1} = \frac{R_{cH}}{3} = \frac{8.64}{3} = 2.88k\Omega,\ X_{mA_1} = \frac{X_{mH}}{3} = \frac{6.84}{3} = 2.28k\Omega,$$
$$\widehat{E}_{A_1} = a\widehat{E}_{A_2}\angle -30^\circ,\ \widehat{I}'_{A_2} = \frac{1}{a}\widehat{I}_{A_2}\angle -30^\circ$$

(**d**) Y-△ **connection**: The nominal values of the line voltages on the primary and the secondary sides are

$$V_{1L} = \sqrt{3}\,V_{a_1n_1} = \sqrt{3} \times 360 = 623.54V$$
$$V_{2L} = V_{a_2b_2} = 120V$$

Thus, the nominal ratings of the three-phase transformer are 2.16kVA 624/120V Y/△ connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$$a = \frac{360}{\frac{120}{\sqrt{3}}} = 5.196,\ R_{A_1} = R_H = 18.9\Omega,\ X_{A_1} = X_H = 21.6\Omega,\ R_{A_2} = \frac{R_L}{3} = \frac{2.1}{3} = 0.7\Omega,$$
$$X_{A_2} = \frac{X_L}{3} = \frac{2.4}{3} = 0.8\Omega,\ R_{cA_1} = R_{cH} = 8.64k\Omega,\ X_{mA_1} = X_{mH} = 6.84k\Omega,\ \widehat{E}_{A_1} = a\widehat{E}_{A_2}\angle 30^\circ,$$
$$\widehat{I}'_{A_2} = \frac{1}{a}\widehat{I}_{A_2}\angle 30^\circ$$

Example 4.6: Three single-phase transformers, each rated at 12kVA 120/240V 60Hz are connected to form a three-phase step-up Y/△ connection. The parameters of each transformer are $R_H = 133.5m\Omega$, $X_H = 201m\Omega$, $R_L = 39.5m\Omega$, $X_L = 61.5m\Omega$, $R_{cL} = 240\Omega$, $X_{mL} = 290\Omega$. What are the nominal voltage, current, and power ratings of the three-phase transformer. When it delivers the rated load at rated voltage and 0.8 pf lagging, determine the line voltages, the line currents, and the efficiency of the three-phase transformer.

Solution: The nominal values of the three-phase transformer are

$$S_{3\phi} = 3S_{1\phi} = 36kVA$$
$$V_{1\phi} = V_{a_1n_1} = 120V$$
$$V_{1L} = \sqrt{3}\,V_{a_1n_1} = \sqrt{3} \times 120 = 208V$$
$$V_{2\phi} = V_{a_2b_2} = 240V$$
$$V_{2L} = V_{a_2b_2} = 240V$$

For the equivalent Y/Y connection, the nominal values of the three-phase transformer are

$$V_{1\phi} = V_{a_1n_1} = 120V$$
$$V_{1L} = \sqrt{3}\,V_{a_1n_1} = \sqrt{3} \times 120 = 208V$$
$$V_{2\phi} = \frac{V_{a_2b_2}}{\sqrt{3}} = \frac{240}{\sqrt{3}} = 138.56V$$
$$V_{2L} = V_{a_2b_2} = 240V$$

Thus, the nominal ratings of the three-phase transformer are 36kVA 208/240V **Y**/△ connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$a = \frac{120}{138.56} = 0.866$, $R_{A_1} = R_L = 39.5m\Omega$, $X_{A_1} = X_L = 61.5m\Omega$, $R_{A_2} = \frac{R_H}{3} = \frac{133.5}{3} = 44.$ $5m\Omega$, $X_{A_2} = \frac{X_H}{3} = \frac{201}{3} = 67m\Omega$, $R_{cA_1} = R_{cL} = 240\Omega$, $X_{mA_1} = X_{mL} = 290\Omega$

Assuming the rated load voltage on a per-phase basis for the equivalent Y/Y connection as the reference, then

$$\widehat{V}_{a_2n_2} = 138.56\angle 0^\circ V$$

For a 0.8 lagging power factor, the load current is

$$\widehat{I}_{A_2} = \frac{S_{1\phi}}{V_{2\phi}}\angle -\cos^{-1}(0.8) = \frac{12000}{138.56}\angle -\cos^{-1}(0.8) = 86.6\angle -36.87^\circ A$$

The per-phase load current in the primary winding is

$$\widehat{I}'_{A_2} = \frac{\widehat{I}_{A_2}}{a}\angle 30^\circ = \frac{86.6}{0.866}\angle(30^\circ - 36.87^\circ) = 100\angle -6.87^\circ A$$

The per-phase voltage induced in the equivalent Y-connected secondary winding is

$$\widehat{E}_{A_2} = \widehat{V}_{a_2n_2} + \widehat{I}_{A_2}(R_{A_2} + jX_{A_2}) = 138.56 + 86.6\angle -36.87^\circ(0.0445 + j0.067)$$
$$= 138.56 + 86.6(0.8 - j0.6)(0.0445 + j0.067)$$
$$= 145.12 + j2.3295 = \sqrt{145.12^2 + 2.3295^2}\angle\tan^{-1}\left(\frac{2.3295}{145.12}\right)\frac{180}{\pi} = 145.147\angle 0.92^\circ V$$

The induced emf in the Y-connected primary winding is

$\widehat{E}_{A_1} = a\widehat{E}_{A_2}\angle 30^\circ = (0.866\angle 30^\circ)(145.147\angle 0.92^\circ) = 0.866 \times 145.$ $147\angle(0.92^\circ + 30^\circ) = 125.7\angle 30.92^\circ V$

The excitation current is

$$\widehat{I}_{\phi A_1} = \frac{\widehat{E}_{A_1}}{\frac{1}{\frac{1}{R_{cA_1}} + \frac{1}{jX_{mA_1}}}} = \widehat{E}_{A_1}\left(\frac{1}{R_{cA_1}} + \frac{1}{jX_{mA_1}}\right) = (125.7\angle 30.92^\circ)\left(\frac{1}{240} + \frac{1}{j290}\right)$$
$$= \left(125.7\cos\left(30.92\frac{\pi}{180}\right) + j125.7\sin\left(30.92\frac{\pi}{180}\right)\right)\left(\frac{1}{240} - j\frac{1}{290}\right) = 0.67204 - j0.10272A$$

Thus, the primary current is

$$\widehat{I}_{A_1} = \widehat{I}_{\phi A_1} + \widehat{I}'_{A_2} = 0.67204 - j0.10272 + 100\angle -6.87^\circ$$
$$= 0.67204 - j0.10272 + 100\left(\cos\left(-6.87\frac{\pi}{180}\right) + j\sin\left(-6.87\frac{\pi}{180}\right)\right) = 99.954 - j12.064A$$

and the primary phase voltage is

$$\widehat{V}_{a_1n_1} = \widehat{E}_{A_1} + \widehat{I}_{A_1}(R_{A_1} + jX_{A_1}) = 125.7\angle 30.92^\circ + (99.954 - j12.064)(0.0395 + j0.0615)$$
$$= \left(125.7\cos\left(30.92\frac{\pi}{180}\right) + j125.7\sin\left(30.92\frac{\pi}{180}\right)\right) + (99.954 - j12.064)(0.0395 + j0.0615)$$
$$= 112.52 + j70.26 = \sqrt{112.52^2 + 70.26^2}\angle\tan^{-1}\left(\frac{70.26}{112.52}\right)\frac{180}{\pi}$$
$$= 132.65\angle 31.982^\circ V$$

The line voltage on the primary side is

$$\widehat{V}_{1L} = \sqrt{3}\,\widehat{V}_{a_1n_1}\angle 30^\circ = 132.65\sqrt{3}\angle 61.982^\circ = 229.76\angle 60.352^\circ V$$

The total input power is

$$P_{in} = 3\,\mathrm{Re}\left(\widehat{V}_{a_1n_1}\widehat{I}^{*}_{A_1}\right) = 3\,\mathrm{Re}((112.52 + j70.26)(99.954 + j12.064)) =$$

$3\,\mathrm{Re}(10399 + j8380.2) = 3 \times 10399 = 31197W$

The total output power is

$$P_o = 3\,\mathrm{Re}\left(\widehat{V}_{a_2b_2}\widehat{I}^{*}_{A_2}\right) = 3\,\mathrm{Re}((138.56\angle 0^\circ)(86.6\angle -36.87^\circ)) =$$

$$3\,\mathrm{Re}(138.56 \times 86.6(0.8 + j0.6))$$

$$= 3\,\mathrm{Re}(9599.4 + j7199.6) = 3 \times 9599.4 = 28798W$$

Hence, the efficiency of the three-phase transformer is

$$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{28798}{31197} \times 100 = 92.3\%$$

Chapter 5 Synchronous Machines

The basic components of a synchronous machine are the stator and rotor. The field winding is placed on the rotor and so called the rotor winding. A DC source is connected to the field winding through slip rings and generates a magnetic flux in the machine. Three-phase armature windings are mounted on the stator spatially displaced by 120 degree electrical from one another and so referred to as the stator windings. A three-phase AC source is applied to the armature winding in a synchronous motor to drive a mechanical load and a three-phase AC power is output from the armature winding in a synchronous generator when it is driven by a prime mover.

Two type of rotors are used in the design of synchronous machines, the cylindrical rotor and a salient-pole rotor.

Balanced three-phase currents generate a rotating magnetic field with a constant magnitude and revolving around the periphery of the rotor at a synchronous speed defined by

$$\omega_s = \frac{4\pi f}{P}$$

where f is the frequency of the AC currents and P is the number of poles in the machine.

The induced voltage of a synchronous machine is directly proportional to the product of the flux ϕ in the machine and the speed ω_s of the machine, that is, $E_a = k\phi\omega_s$.

5.1 Synchronous Generators

A generator is driven by a mechanical source, a prime mover, to turn at the synchronous speed. When a DC source is applied to the field winding, three-phase AC voltages are induced in the armature windings.

5.1.1 Synchronous Generators with a Cylindrical Rotor

Figure 5.1 shows an equivalent circuit for a synchronous generator with a cylindrical rotor, where R_f, L_f, V_f, and I_f are the field resistance, inductance, voltage, and current, X_s is the synchronous reactance, R_a, $\hat{V}_a$, and $\hat{I}_a$ are the armature resistance, voltage, and current, $\hat{E}_a$ is the generated voltage (induced voltage). It follows from KVL that

$$\hat{E}_a = \hat{V}_a + \hat{I}_a R_a + j\hat{I}_a X_s$$

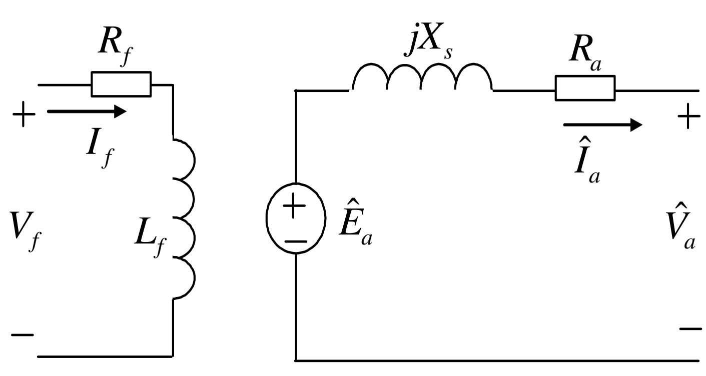


Figure 5.1 The per-phase equivalent circuit of a

Figure 5.2 shows the phasor diagram for a synchronous generator with a lagging load. θ is the power factor angle and δ is the torque angle. The torque angle is negative for the synchronous generator.

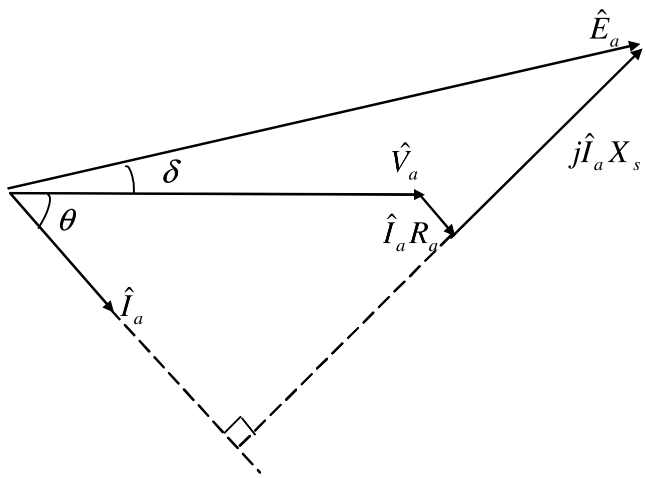


Figure 5.2 The phasor diagram for a generator

1. **The resistance Test**: This test is conducted to measure the armature winding resistance of a synchronous generator by measuring the resistance from line to line, R_L, when it is at rest and the field winding is open. If the generator is Y-connected, the per-phase resistance is $R_a = 0.5R_L$. On the other hand, for a $\triangle$-connected generator, $R_a = 1.5R_L$.

2. **The Open-Circuit Test** (**No-Load Test**): This test is performed by driving the generator at its rated speed while the armature winding is left open. The open-circuit voltage between any two pair of terminals of the armature windings is recorded when the field current is varied from zero to its rated value. The graph of the per-phase open-circuit voltage versus the field current is referred to as the open-circuit characteristic of a generator.

3. **The Short-Circuit Test**: This tested is carried out by driving the generator at its rated speed when the terminals of the armature winding are shorted. The line current of the armature winding is recorded when the field current is gradually increased. The graph of the per-phase short-circuit current versus the field current is called the short-circuit characteristic of a generator.

4. **Calculation of the Synchronous Reactance**: Let I_{fr} be the field current which gives the rated per-phase voltage (V_{aoc}) from the open-circuit test and I_{asc} be the phase current corresponding to the field current I_{fr} from the short-circuit test. Then, the synchronous reactance is calculated by

$|Z_s| = \frac{V_{aoc}}{I_{asc}}$ (Synchronous impedance $Z_s = R_a + jX_s$)

$X_s = \sqrt{|Z_s|^2 - R_a^2}$

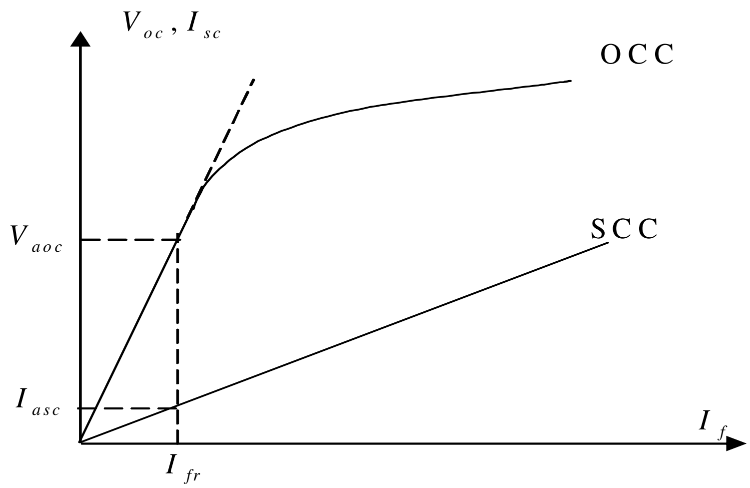


Figure 5.3 OCC and SCC of a synchronous machine

5. **The Developed Torque and Efficiency**:

The output power of a synchronous generator is

$$P_o = 3V_a I_a \cos\theta$$

The copper loss in the armature winding is

$$P_{cu} = 3I_a^2 R_a$$

The developed power is

$$P_d = P_o + P_{cu} = 3V_a I_a \cos\theta + 3I_a^2 R_a$$

If $R_a = 0$, then

$$P_d = \frac{3V_a E_a \sin\delta}{X_s}$$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_s}$$

The input power to the field winding is

$$P_f = V_f I_f$$

If P_r is the rotational loss and P_{stray} is the stray loss, then the input power $P_{in} (= P_{mech} + P_f)$ is

$$P_{in} = P_o + P_{cu} + P_r + P_{stray} + P_f = 3V_a I_a \cos\theta + 3I_a^2 R_a + P_r + P_{stray} + P_f$$

The core loss $P_c = P_r + P_{stray} + P_f$ does not change much with the load change and can be considered as constant. The efficiency of the generator

$$\eta = \frac{P_o}{P_{in}} = \frac{3V_a I_a \cos\theta}{3V_a I_a \cos\theta + 3I_a^2 R_a + P_c}$$

reaches its maximum when

$$3I_a^2 R_a = P_c$$

As a result, the generator reaches its maximum efficiency when the load current is

$$I_a = \sqrt{\frac{P_c}{3R_a}}$$

6. **The Voltage Regulation**: The voltage regulation of a synchronous generator is defined as the ratio of the change in the terminal voltage from no load to full load, that is,

$$VR\% = \frac{V_{aNL} - V_{aFL}}{V_{aFL}}$$

where V_{aNL} and V_{aFL} are the no-load voltage and the full-load voltage of the synchronous generator.

Example 5.1: A 500kVA, 2300V, three-phase, Y-connected, synchronous generator is operated at its rated speed to obtain its rated no-load voltage. When a short-circuit is established, the phase current is 150A. The average resistance of each phase is 0.5Ω. The core loss is assumed to be 20kW. Determined the synchronous reactance per phase. Calculate the efficiency and voltage regulation when the generator delivers the rated load at its rated voltage and 0.8 pf lagging.

Solution:

The open-circuit phase voltage is $V_{aoc} = 2300/\sqrt{3} = 1327.9V$

The short-circuit phase current is $I_{asc} = 150A$

Therefore the synchronous impedance is $|Z_s| = \frac{V_{aoc}}{I_{asc}} = \frac{1327.9}{150} = 8.85\Omega$

Thus the synchronous reactance is $X_s = \sqrt{|Z_s|^2 - R_a^2} = \sqrt{8.85^2 - 0.5^2} = 8.84\Omega$

The rated phase voltage is $V_a = \frac{2300}{\sqrt{3}} = 1327.9V$ and it is assumed that $\widehat{V}_a = 1327.9\angle 0^\circ V$

The rated load current is $I_a = \frac{500000}{3\times 1327.9} = 125.51A$ and $\widehat{I}_a = 125.51\angle -36.87^\circ A$

It follows from the per-phase equivalent circuit that

$$\widehat{E}_a = \widehat{V}_a + \widehat{I}_a Z_s = 1327.9\angle 0^\circ + 125.51\angle -36.87^\circ(0.5 + j8.84)$$
$$= 1327.9 + 125.51(0.8 - j0.6)(0.5 + j8.84) = 2043.8 + j849.95$$
$$= \sqrt{2043.8^2 + 849.95^2}\angle \tan^{-1}\left(\frac{849.95}{2043.8}\right)\frac{180}{\pi} = 2213.5\angle 22.6^\circ V$$

Thus, the no-load phase voltage is $V_{aNL} = E_a = 2213.5V$ and the full-load phase voltage is $V_{aFL} = V_a = 1327.9V$, which implies that

$$VR\% = \frac{2213.5 - 1327.9}{1327.9} \times 100 = 66.7\%$$

The output power of the synchronous generator is

$$P_o = 3V_a I_a \cos\theta = 3 \times 1327.9 \times 125.51 \times 0.8 = 400000W$$

The copper loss in the armature winding is

$$P_{cu} = 3I_a^2 R_a = 3 \times 125.51^2 \times 0.5 = 23629W$$

The input power is

$$P_{in} = P_o + P_{cu} + P_c = 400000 + 23629 + 20000 = 443630W$$

Thus, the efficiency of the generator is

$$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{400000}{443630} \times 100 = 90.2\%$$

5.1.2 Synchronous Generators with a Salient-Pole Rotor

Unlike a cylindrical rotor synchronous generator, a salient-pole synchronous generator has a large air-gap in the region between the poles than in the region just above the poles, as is evidenced from Figure 5.3. Therefore, the reluctances of the two regions in a salient-pole generator differ significantly.

In order to account for this difference, the synchronous reactance is split into two reactances. The component of the synchronous reactance along the pole-axis (the d-axis) is called the direct-axis synchronous reactance X_d and the other component along the axis between the poles (the q-axis) is referred to as the quadrature-axis synchronous reactance X_q.

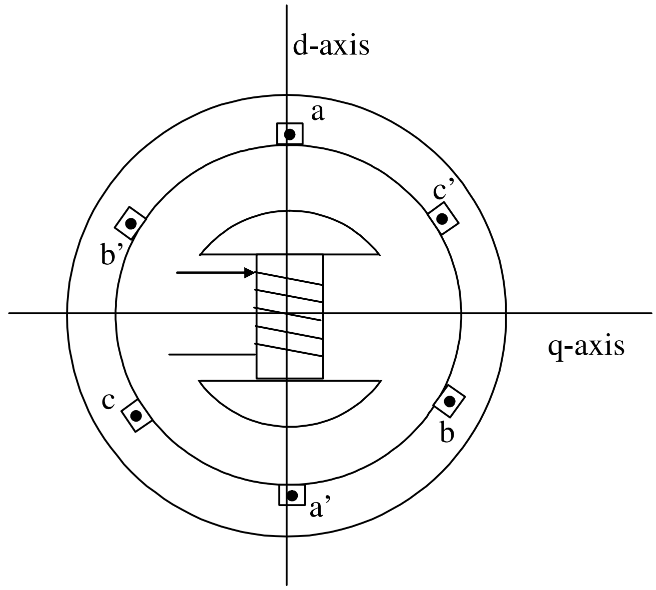


Figure 5.4 A salient-pole synchronous

The armature current $\hat{I}_a$ is also resolved into two components: the direct-axis component $\hat{I}_d$ and quadrature-axis component $\hat{I}_q$. Then, $\hat{I}_a = \hat{I}_d + \hat{I}_q$. The direct-axis component $\hat{I}_d$ produces the field along the d-axis and lags $\hat{E}_a$ by 90° and the quadrature-axis component $\hat{I}_q$ produces the field along the q-axis and is in phase with $\hat{E}_a$.

Let $\hat{E}_a$ be the per-phase generated voltage under no-load and $\hat{E}_d$ and $\hat{E}_q$ be the induced emfs in the armature winding by the currents $\hat{I}_d$ and $\hat{I}_q$, respectively. Then $\hat{E}_d$ and $\hat{E}_q$ can be expressed in terms of X_d and X_q as

$\hat{E}_d = -j\hat{I}_d X_d$ and $\hat{E}_q = -j\hat{I}_q X_q$

The per-phase terminal voltage of the generator is

$$\hat{V}_a = \hat{E}_a + \hat{E}_d + \hat{E}_q - \hat{I}_a R_a = \hat{E}_a - j\hat{I}_d X_d - j\hat{I}_q X_q - \hat{I}_a R_a$$
$$= \hat{E}_a - j\hat{I}_d X_d - j\left(\hat{I}_a - \hat{I}_d\right)X_q - \hat{I}_a R_a = \hat{E}_a - j\hat{I}_d(X_d - X_q) - j\hat{I}_a X_q - \hat{I}_a R_a$$
$$= \hat{E}_a' - j\hat{I}_a X_q - \hat{I}_a R_a$$

where $\hat{E}_a' = \hat{E}_a - j\hat{I}_d(X_d - X_q)$, as shown in Figure 5.4.

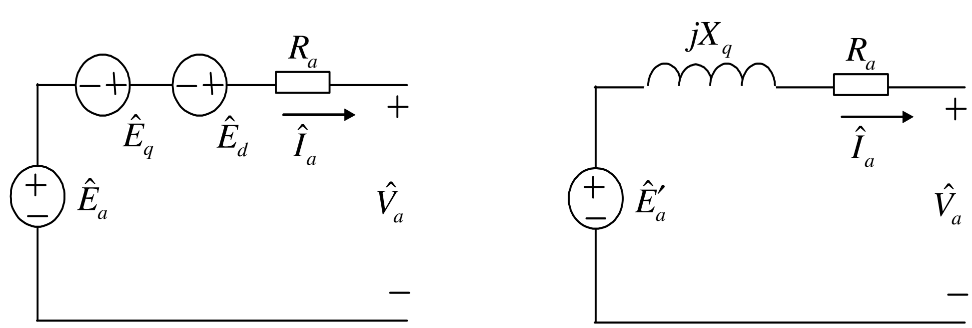


Figure 5.5 Equivalent circuits of a salient-pole synchronous generator

The phasor diagrams for a lagging load and a leading load are shown in Figure 5.5.

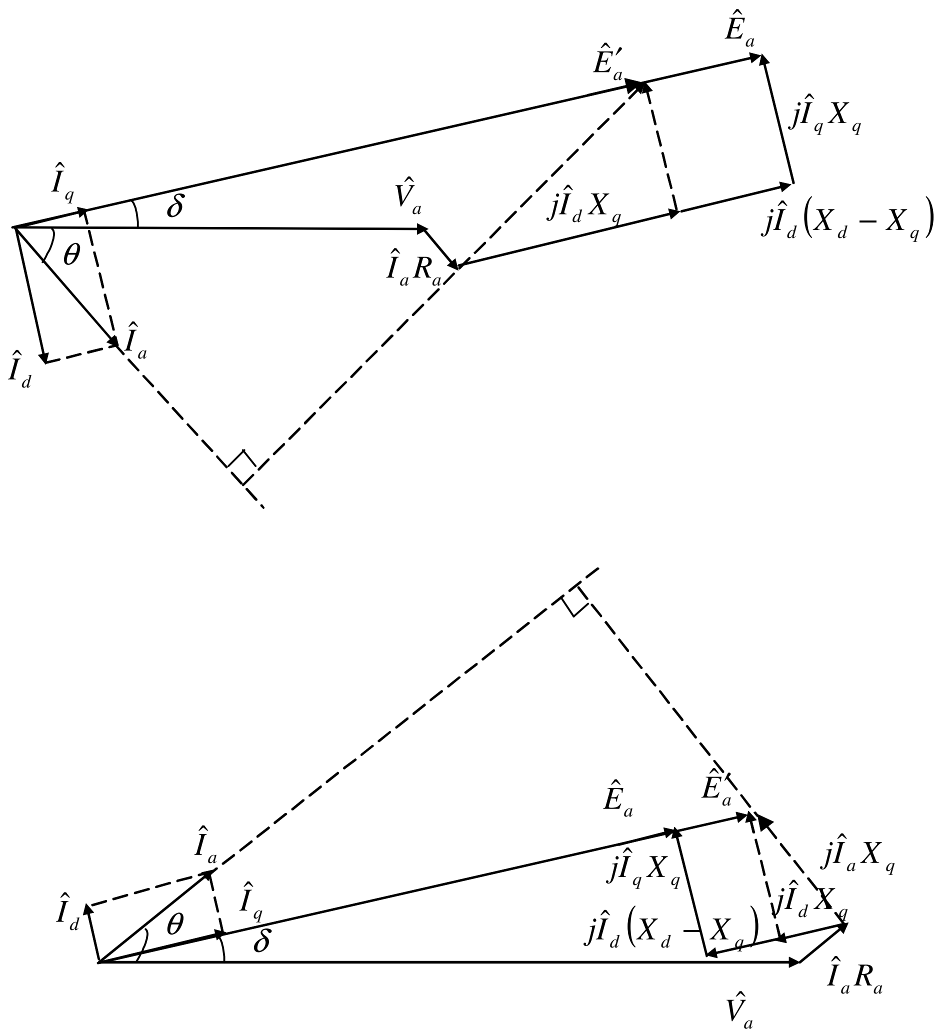


Figure 5.6 Phasor diagram of a salient-pole synchronous

The output power is

$$P_o = 3\,\mathrm{Re}\left(\hat{V}_a \hat{I}_a^*\right) = 3V_a I_a \cos\theta$$

The copper loss in the armature winding is

$$P_{cu} = 3I_a^2 R_a$$

The developed power is

$$P_d = P_o - P_{cu} = 3V_a I_a \cos\theta - 3I_a^2 R_a$$

If $R_a = 0$, then

$$P_d = \frac{3V_a E_a \sin|\delta|}{X_d} + \frac{3(X_d - X_q)}{2X_d X_q} V_a^2 \sin|2\delta|$$

and the developed torque is

$$\tau_d = \frac{3V_a E_a \sin|\delta|}{X_d \omega_s} + \frac{3(X_d - X_q)}{2X_d X_q \omega_s} V_a^2 \sin|2\delta|$$

The input power to the field winding is

$$P_f = V_f I_f$$

If P_r is the rotational loss and P_{stray} is the stray loss, then the input power $P_{in} (= P_{mech} + P_f)$ is

$P_{in} = P_o + P_{cu} + P_r + P_{stray} + P_f = 3V_aI_a\cos\theta + 3I_a^2R_a + P_r + P_{stray} + P_f = 3V_aI_a\cos\theta + 3I_a^2R_a + P_c$
where $P_c = P_r + P_{stray} + P_f$ is the core loss. The efficiency of the generator

$$\eta = \frac{P_o}{P_{in}} = \frac{3V_aI_a\cos\theta}{3V_aI_a\cos\theta + 3I_a^2R_a + P_c}$$

Example 5.2: A 70MVA 13.8kV 60Hz two-pole Y-connected three-phase salient-pole synchronous generator has $R_a = 0$, $X_d = 1.83\Omega$, and $X_q = 1.21\Omega$. It delivers the rated load at 0.8 pf lagging. Determine δ, $\widehat{E}_a$, $VR\%$, P_d, and τ_d.

Solution: The phase terminal voltage is

$$\widehat{V}_a = \frac{13800}{\sqrt{3}}\angle 0^\circ = 7967.4\angle 0^\circ V$$

The phase load current is

$$\widehat{I}_a = \frac{70\times 10^6}{\sqrt{3}\times 13800}\angle -36.87^\circ = 2928.6\angle -36.87^\circ A$$

It follows from the equivalent circuit that

$$\widehat{E}'_a = \widehat{V}_a + j\widehat{I}_aX_q + \widehat{I}_aR_a = 7967.4\angle 0^\circ + (2928.6\angle -36.87^\circ)(j1.21)$$
$$= 7967.4 + 2928.6(0.8 - j0.6)(j1.21) = 10094 + j2834.9$$
$$= \sqrt{10094^2 + 2834.9^2}\angle \tan^{-1}\left(\frac{2834.9}{10094}\right)\frac{180}{\pi} = 10485\angle 15.7^\circ V$$

The torque angle is 15.7°. The d- and q-axis currents are

$$\widehat{I}_d = I_a\sin(|\theta| + \delta)\angle(-90^\circ + \delta)$$
$$= 2928.6\sin\left((36.87 + 15.7)\frac{\pi}{180}\right)\angle(-90^\circ + 15.7^\circ) = 2323.6\angle -74.3^\circ A$$
$$\widehat{I}_q = I_a\cos(|\theta| + \delta)\angle\delta = 2928.6\cos\left((36.87 + 15.7)\frac{\pi}{180}\right)\angle 15.7^\circ = 1780\angle 15.7^\circ A$$

The generated voltage is

$$\widehat{E}_a = \widehat{E}'_a + j\widehat{I}_d(X_d - X_q) = 10094 + j2834.9 + (2323.6\angle -74.3^\circ)(j(1.83 - 1.21))$$
$$= 10094 + j2834.9 + 2323.6\left(\cos\left((-74.3)\frac{\pi}{180}\right) + j\sin\left((-74.3)\frac{\pi}{180}\right)\right)(j(1.83 - 1.21))$$
$$= 11481 + j3224.7 = \sqrt{11481^2 + 3224.7^2}\angle \tan^{-1}\left(\frac{3224.7}{11481}\right)\frac{180}{\pi} = 11925\angle 15.7^\circ V$$

or is given by

$$\widehat{E}_a = \widehat{V}_a + j\widehat{I}_dX_d + j\widehat{I}_qX_q + \widehat{I}_aR_a$$
$$= 7967.4\angle 0^\circ + (2323.6\angle -74.3^\circ)(j1.83) + (1780\angle 15.7^\circ)(j1.21) + 0$$
$$= 7967.4 + 2323.6\left(\cos\left((-74.3)\frac{\pi}{180}\right) + j\sin\left((-74.3)\frac{\pi}{180}\right)\right)(j1.83)$$
$$+ 1780\left(\cos\left((15.7)\frac{\pi}{180}\right) + j\sin\left((15.7)\frac{\pi}{180}\right)\right)(j1.21)$$
$$= 11478 + j3224.1 = \sqrt{11478^2 + 3224.1^2}\angle \tan^{-1}\left(\frac{3224.1}{11478}\right)\frac{180}{\pi} = 11922\angle 15.7^\circ V$$

The developed power is

$$P_d = P_o - P_{cu} = P_o = 3V_aI_a\cos\theta = 3\times 7967.4\times 2928.6\times 0.8 = 5.6\times 10^7 W$$

The synchronous speed is

$$\omega_s = \frac{2\pi f}{P} = \frac{2\pi\times 60}{2} = 188.5 rad/s$$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_s} = \frac{5.6\times 10^7}{188.5} = 2.9708\times 10^5 N\bullet m$$

The voltage regulation is

$$VR\% = \frac{11925 - 7967.4}{7967.4}\times 100 = 49.7\%$$

5.2 Synchronous Motors

A synchronous motor is powered by a electrical source to drive a load at the synchronous speed. When a DC source is applied to the field winding and three-phase AC voltages are connected to the armature windings, the motor will turns its load at the synchronous speed.

5.2.1 Synchronous Motors with a Cylindrical Rotor

The equivalent circuit for a cylindrical-rotor synchronous motor is shown in Figure 5.7, which is the same as the cylindrical-rotor synchronous generator with the reversed armature current direction. It follows from Kirchhoff's voltage law that

$$\widehat{V}_a = \widehat{E}_a + \widehat{I}_a R_a + j\widehat{I}_a X_s$$

Figure 5.8 shows the phasor diagrams for a synchronous motor with a lagging load. θ is the power factor angle and δ is the torque angle. The torque angle is negative for the synchronous motor.

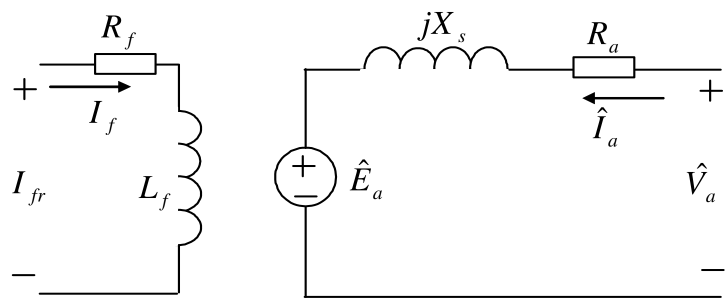


Figure 5.7 The per-phase equivalent circiut of a motor

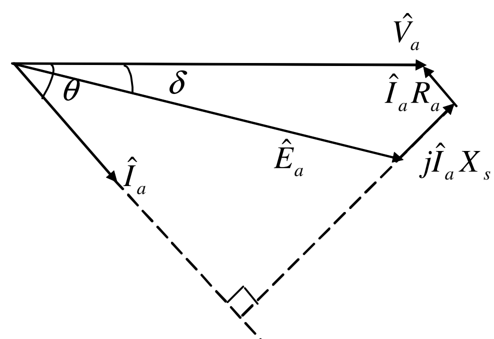


Figure 5.8 The phasor diagram

The input power of a synchronous motor is

$$P_{in} = 3V_a I_a \cos\theta + V_f I_f$$

The copper loss is

$$P_{cu} = 3I_a^2 R_a + V_f I_f$$

The developed power is

$$P_d = P_{in} - P_{cu} = 3V_a I_a \cos\theta - 3I_a^2 R_a$$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_s}$$

If $R_a = 0$, then

$$P_d = \frac{3V_a E_a \sin\delta}{X_s}$$

If P_r is the rotational loss and P_{stray} is the stray loss, then the output power $P_o (= \tau_o \omega_s)$ is

$$P_o = P_d - P_r - P_{stray}$$

The efficiency of the motor is

$$\eta = \frac{P_o}{P_{in}}$$

Example 5.3: A 220V 60Hz 3-phase 2-pole Y-connected synchronous motor has a synchronous impedance of 0.25+j2.5Ω/phase. The motor delivers the rated load of 80A at 0.707 pf leading. Determine (a) the generated voltage, (b) the torque angle, (c) the power developed by the motor, and (d) the developed torque.

Solution: The phase voltage is $V_a = \frac{220}{\sqrt{3}} = 127V$. Assume $\hat{V}_a = 127\angle 0^\circ V$. The phase armature current is $\hat{I}_a = 80\angle 45^\circ A$. It follows from the per-phase equivalent circuit that

$$\hat{E}_a = \hat{V}_a - \hat{I}_a R_a - j\hat{I}_a X_s = 127\angle 0^\circ - (80\angle 45^\circ)(0.25 + j2.5)$$

$$= 127 - 80(0.701 + j0.707)(0.25 + j2.5) = 254.38 - j154.34$$

$$= \sqrt{254.38^2 + 154.34^2}\angle \tan^{-1}\left(\frac{-154.34}{254.38}\right)\frac{180}{\pi} = 297.54\angle -31.2^\circ V$$

The torque angle is -31.2°.

$$P_d = P_{in} - P_{cu} = 3V_a I_a \cos\theta + P_f - (3I_a^2 R_a + P_f) = 3 \times (127 \times 80 \times 0.707 - 80^2 \times 0.25) = 16749W$$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_s} = \frac{16749}{\frac{4\pi\times 60}{2}} = 44.428N \cdot m$$

5.2.2 Synchronous Motors with a Salient-Pole Rotor

Similar to a salient-pole synchronous generator, the per-phase equivalent circuit for a salient-pole synchronous motor is required to analyse the motor performance, which is shown in Figure 5.9.

The per-phase terminal voltage of the motor is

$$\hat{V}_a = \hat{E}_a - \hat{E}_d - \hat{E}_q + \hat{I}_a R_a = \hat{E}_a + j\hat{I}_d X_d + j\hat{I}_q X_q + \hat{I}_a R_a$$

$$= \hat{E}_a + j\hat{I}_d X_d + j\left(\hat{I}_a - \hat{I}_d\right)X_q + \hat{I}_a R_a = \hat{E}_a + j\hat{I}_d(X_d - X_q) + j\hat{I}_a X_q + \hat{I}_a R_a$$

$$= \hat{E}_a' + j\hat{I}_a X_q + \hat{I}_a R_a$$

where $\hat{E}_a' = \hat{E}_a + j\hat{I}_d(X_d - X_q)$, as shown in Figure 5.9. The phasor diagrams for a leading load and a lagging load are shown in Figure 5.10.

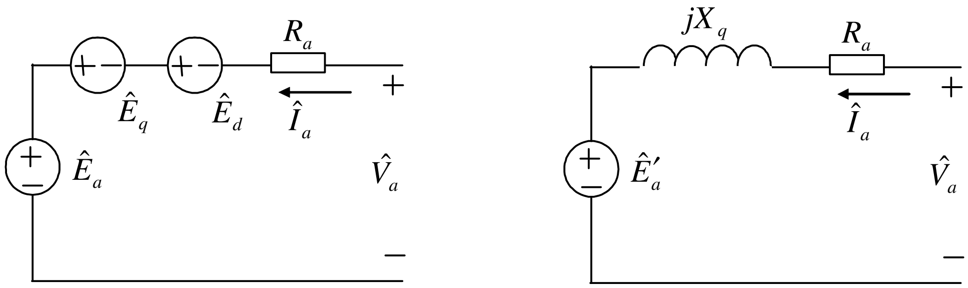


Figure 5.9 The equivalent circiut of a salient-pole motor

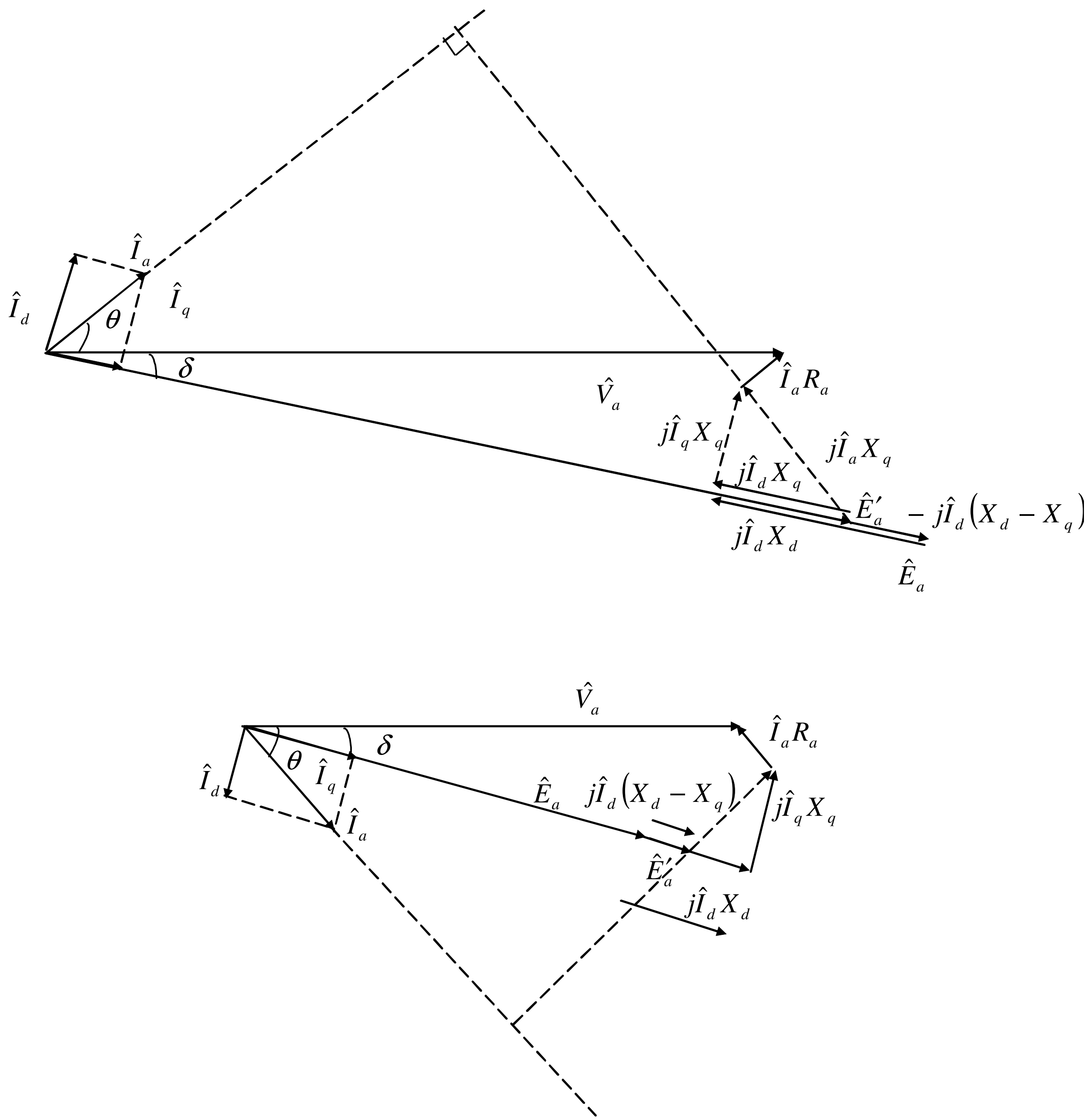


Figure 5.10 The Phasor diagram

Then the input power is

$$P_{in} = 3\,\mathrm{Re}\left(\widehat{V}_a \widehat{I}_a^*\right) = 3V_a I_a \cos\theta + V_f I_f$$

The copper loss is

$$P_{cu} = 3I_a^2 R_a + V_f I_f$$

The developed power is

$$P_d = P_{in} - P_{cu} = 3V_a I_a \cos\theta - 3I_a^2 R_a$$

If $R_a = 0$, then

$$P_d = \frac{3V_a E_a \sin|\delta|}{X_d} + \frac{3(X_d - X_q)}{2X_d X_q} V_a^2 \sin|2\delta|$$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_s}$$

If P_r is the rotational loss and P_{stray} is the stray loss, then the output power $P_o(= \tau_o\omega_s)$ is

$$P_o = P_d - P_r - P_{stray}$$

The efficiency of the motor

$$\eta = \frac{P_o}{P_{in}}$$

Example 5.4: A 208V 60Hz three-phase Y-connected salient-pole synchronous motor operates at full load and draws a current of 40A at 0.8 pf lagging. The d- and q-axis reactances are 2.7$\Omega/phase$ and 1.7$\Omega/phase$, respectively. The armature-winding resistance is negligible, and the rotational loss is 5% of the power developed by the motor. Determine (a) the developed voltage, (b) the developed power, and (c) the efficiency.

Solution: The per-phase load voltage and current are $\hat{V}_a = \frac{208}{\sqrt{3}}\angle 0^\circ = 120\angle 0^\circ V$ and $\hat{I}_a = 40\angle - \cos^{-1}(0.8) = 40\angle - 36.87^\circ A$.

It follows from the per-phase equivalent circuit that

$$\hat{E}'_a = \hat{V}_a - j\hat{I}_aX_q - \hat{I}_aR_a = 120\angle 0^\circ - (40\angle - 36.87^\circ)(j1.7) = 120 - 40(0.8 - j0.6)(j1.7)$$
$$= 79.2 - j54.4 = \sqrt{79.2^2 + 54.4^2}\angle \tan^{-1}\left(\frac{-54.4}{79.2}\right)\frac{180}{\pi} = 96.083\angle - 34.48^\circ V$$

which means that the torque angle is $\delta = -34.48^\circ$. It follows from the phasor diagram that the absolute value of the angle between $\hat{I}_q$ and $\hat{I}_a$ are $\alpha = |\theta - \delta| = |-36.87 - (-34.48)| = 2.39^\circ$. Therefore, the d-axis armature current is

$$\hat{I}_d = I_a \sin\alpha\angle(\delta - 90^\circ) = 40\sin\left(2.39\frac{\pi}{180}\right)\angle(-34.48 - 90)^\circ = 1.668\angle - 124.48^\circ$$
$$= 1.668\left(\cos\left(-124.48\frac{\pi}{180}\right) + j\sin\left(-124.48\frac{\pi}{180}\right)\right) = -0.944 - j1.375A$$

(a) The per-phase developed voltage is

$$\hat{E}_a = \hat{E}'_a - j\hat{I}_d(X_d - X_q) = (79.2 - j54.4) - j(-0.944 - j1.375)(2.7 - 1.7)$$
$$= 77.825 - j53.456 = \sqrt{77.825^2 + 53.456^2}\angle \tan^{-1}\left(\frac{-53.456}{77.825}\right)\frac{180}{\pi} = 94.415\angle - 34.48^\circ V$$

(b) As $R_a = 0$, the AC input power is the same as the developed power, that is,

$$P_d = P_{in} - P_{cu} = P_{in} = 3V_aI_a\cos\theta = 3 \times 120 \times 40 \times 0.8 = 11520W$$

Or it can be calculated by

$$P_d = \frac{3V_aE_a\sin|\delta|}{X_d} + \frac{3(X_d - X_q)}{2X_dX_q}V_a^2\sin|2\delta|$$
$$= \frac{3\times120\times94.415\times\sin\left(34.48\frac{\pi}{180}\right)}{2.7} + \frac{3(2.7-1.7)}{2\times2.7\times1.7} \times 120^2 \times \sin\left(2 \times 34.48\frac{\pi}{180}\right) = 11519W$$

(c) The output power is

$$P_o = P_{in} - P_{cu} - P_r - P_{stray} = P_{in} - P_r = 11520 - 0.05 \times 11520 = 10944W$$

and the efficiency is

$$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{10944}{11520} \times 100 = 95\%$$

Chapter 6 Induction Motors

6.1 Three-Phase Induction Motors

The essential components of an induction motor are a stator and a rotor. A balanced three-phase winding is placed on the stator. There are two types of rotors: a squirrel-cage rotor and a wound rotor. Rotor windings are short-circuited for both types of rotors. When the stator winding of a three-phase induction motor is connected to a three-phase power supply, it produces a rotating magnetic field which is constant in magnitude and revolves at the synchronous speed given by

$$\omega_s = \frac{4\pi f}{P} \text{ or } N_s = \frac{120f}{P}$$

where f is the frequency of the power supply and P is the number of poles. This rotating magnetic field induces emf in the rotor winding. Since the rotor winding is short-circuited, the induced emf produces an induced current in the rotor winding, which, together with the rotating magnetic field, induces torque on the rotor winding to make the rotor spin at speed ω_m. It is important to note that the induced voltage is proportional to the relative speed of the rotor with respect to the synchronous speed of the rotating magnetic field. Such a relative speed is defined as the slip speed

$$\omega_r = \omega_s - \omega_m \text{ or } N_r = N_s - N_m$$

and the ratio between the relative speed and the synchronous speed is referred to as the slip

$$s = \frac{\omega_s - \omega_m}{\omega_s} = \frac{N_s - N_m}{N_s}$$

The motor speed can be expressed as

$$\omega_m = (1 - s)\omega_s \text{ or } N_m = (1 - s)N_s$$

The frequency of the induced voltage in the rotor is

$$f_r = \frac{PN_r}{120} = \frac{P(N_s - N_m)}{120} = \frac{PN_s}{120}\frac{N_s - N_m}{N_s} = sf$$

When the rotor is stationary, the slip is 1 and the rotor appears exactly like a short-circuited secondary winding of a transformer. Therefore, an induction motor is a transformer with a rotating secondary winding and the equivalent circuit for a transformer can be used for an induction motor.

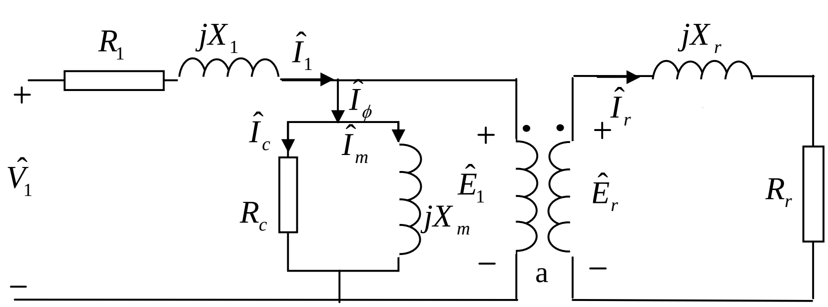


Figure 6.1

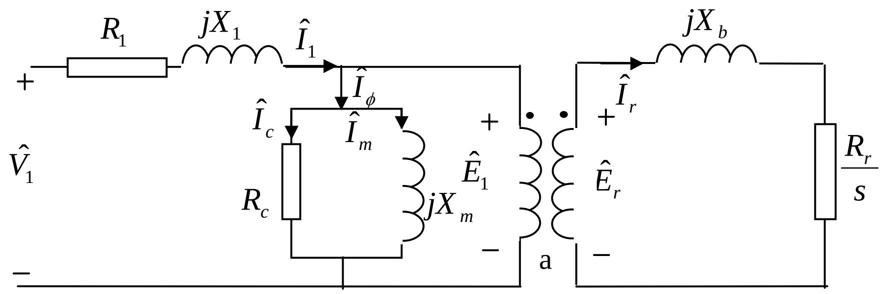


Figure 6.2

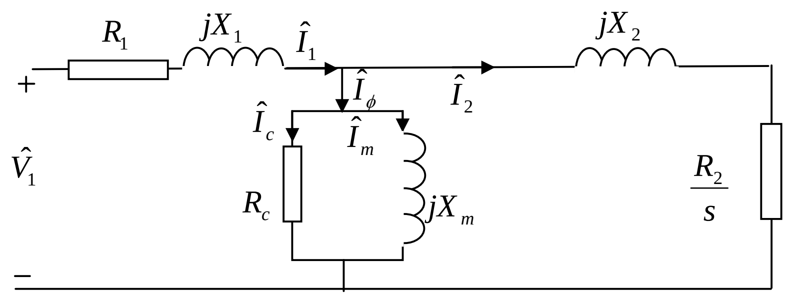


Figure 6.3

Figure 6.1 shows a per-phase equivalent circuit for a three-phase induction motor, where

$\hat{V}_1$ =per-phase stator voltage

$\hat{I}_1$ =per-phase stator current

R_1 =per-phase stator resistance

L_1 =per-phase stator leakage inductance

$X_1 = 2\pi f L_1$ =per-phase stator leakage reactance

$\hat{I}_r$ =per-phase rotor current

R_r =per-phase rotor resistance

L_r =per-phase rotor leakage inductance

$X_b = 2\pi f L_r$ =per-phase rotor leakage reactance at s=1

$X_r = 2\pi f_r L_r = sX_b$ =per-phase rotor leakage reactance

X_m =per-phase magnetization reactance

R_c =per-phase core-loss resistance

$\hat{E}_1$ =per-phase induced voltage in the stator winding

$\hat{E}_b$ =per-phase induced voltage in the rotor winding at s=1

$\hat{E}_r = s\hat{E}_b$ =per-phase induced voltage in the rotor winding

$\hat{I}_\phi = \hat{I}_c + \hat{I}_m$ =per-phase excitation current

$\hat{I}_c$ =per-phase core-loss current

$\hat{I}_m$ =per-phase magnetization current

a =effective turns ratio

Note that

$$\hat{I}_r = \frac{\hat{E}_r}{R_r + jX_r} = \frac{s\hat{E}_b}{R_r + jsX_b} = \frac{\hat{E}_b}{\frac{R_r}{s} + jX_b}$$

Hence the equivalent circuit Figure 6.1 can be modified as Figure 6.2. Referring the rotor side to the stator side, the equivalent circuit Figure 6.2 is transformed to the equivalent circuit Figure 6.3. The approximate per-phase equivalent circuit is given as Figure 6.4.

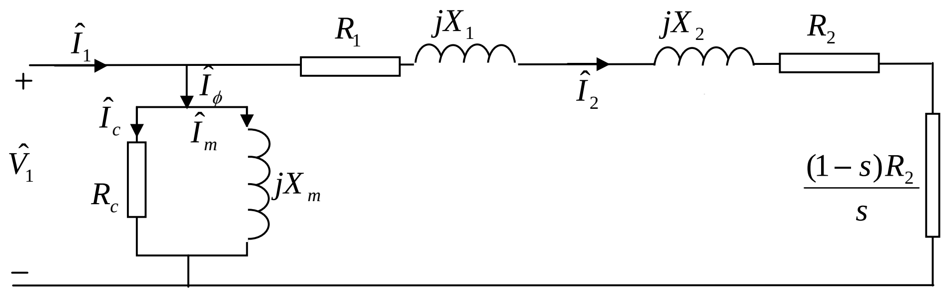


Figure 6.4

The Stator Resistance Test

Let R_L be the DC value of the resistance between any two terminals of the motor. Then the per-phase resistance is

$R_1 = 0.5R_L$ for Y-connection

$R_1 = 1.5R_L$ for $\triangle$-connection

The Blocked-Rotor Test

The rotor is held still by external torque and a variable three-phase power is applied to the stator winding. The stator voltage is carefully increased from zero until the motor draws the rated current. Let V_{br}, I_{br}, and P_{br} be the input voltage, current, and power on a per-phase basis. Then,

$$R_e = \frac{P_{br}}{I_{br}^2}$$

$$|Z_e| = \frac{V_{br}}{I_{br}^2}$$

where $Z_e = R_e + jX_e = R_1 + R_2 + j(X_1 + X_2)$. Therefore,

$$R_2 = R_e - R_1$$

$$X_e = \sqrt{|Z_e|^2 - R_e^2}$$

For all practical purposes, it is assumed that $X_1 = X_2 = 0.5X_e$

The No-Load Test

The rated voltage is applied to the stator winding and the motor operates without any load. Let V_{NL}, I_{NL}, and P_{NL} be the input voltage, current, and power on a per-phase basis. Let P_r be the rotational loss on a per-phase basis. Then the power loss in R_c is

$$P_c = P_{NL} - P_r$$

and

$$R_c = \frac{V_{NL}^2}{P_c}$$

$$|Z_\phi| = \frac{V_{NL}}{I_{NL}}$$

where $Z_\phi = \frac{1}{\frac{1}{R_c}+\frac{1}{jX_m}}$. Note that $\left|\frac{1}{Z_\phi}\right|^2 = \left(\frac{1}{R_c}\right)^2 + \left(\frac{1}{X_m}\right)^2$. As a result, we have

$$X_m = \frac{1}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2 - \left(\frac{1}{R_c}\right)^2}}$$

Power Flow Diagram

The following are based on the exact per-phase equivalent circuit.

The input power: $P_{in} = 3V_1 I_1 \cos\theta$

The stator copper loss: $P_{scu} = 3I_1^2 R_1$

The air-gap power: $P_{ag} = P_{in} - P_{scu} = 3I_2^2 \frac{R_2}{s}$ (the power consumed by $\frac{R_2}{s}$)

The rotor copper loss: $P_{rcu} = 3I_2^2 R_2 = sP_{ag}$

The developed power: $P_d = P_{ag} - P_{rcu} = P_{ag} - sP_{ag} = (1-s)P_{ag} = 3I_2^2 \frac{(1-s)R_2}{s}$

The rotational loss: $P_r = P_c + P_{fw} + P_{stray}$

The output power: $P_o = P_d - P_r$

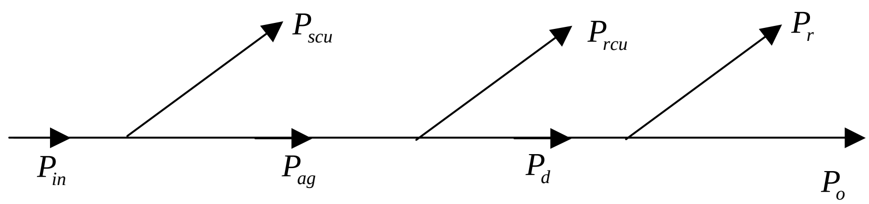


Figure 6.5

The following are based on the approximate equivalent circuit.

It follows from the approximate equivalent circuit Figure 6.4 that the rotor current is

$$\hat{I}_2 = \frac{\hat{V}_1}{R_1+R_2+j(X_1+X_2)+\frac{(1-s)R_2}{s}} = \frac{V_1}{\sqrt{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}}\angle\theta_2$$

So the developed power is

$$P_d = 3I_2^2\frac{(1-s)R_2}{s} = \frac{3V_1^2\frac{(1-s)R_2}{s}}{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}$$

which is a function of s. By differentiating P_d with respect to s and setting the derivative to zero, we can find the slip for the maximum power, which is given by

$$s_{\max,p} = \frac{R_2}{R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}$$

and

$$P_{d,\max} = \frac{3}{2}\frac{V_1^2}{R_1+R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}$$

The Efficiency

Using the approximate equivalent circuit, the output power can also be calculated by

$P_o = P_d - P_r = P_{ag} - P_{rcu} - P_r = P_{in} - P_{scu} - P_{rcu} - P_r = 3V_1 I_2 \cos\theta - 3I_2^2 R_1 - 3I_2^2 R_2 -$

Therefore, the efficiency is

$$\eta = \frac{P_o}{P_{in}} = \frac{3V_1 I_2 \cos\theta - 3I_2^2(R_1+R_2) - P_r}{3V_1 I_2 \cos\theta}$$

Differentiating η with respect to I_2 and setting the derivative equal to zero gives

$$3I_2^2(R_1 + R_2) = P_r$$

which implies that the efficiency is maximum when the sum of the stator and the rotor cooper losses is equal to the rotational loss, that is,

$$\eta_{\max} = \frac{3V_1 I_2 \cos\theta - 2P_r}{3V_1 I_2 \cos\theta}$$

at

$$I_{2,\max,\eta} = \sqrt{\frac{P_r}{3I_2(R_1+R_2)}}$$

The Developed Torque

Note that $P_d = \omega_m \tau_d$. Thus the developed torque is given by

$$\tau_d = \frac{P_d}{\omega_m} = \frac{\dfrac{3V_1^2 \frac{(1-s)R_2}{s}}{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}}{\omega_m} = \frac{3V_1^2 \frac{(1-s)R_2}{s}}{(1-s)\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]} = \frac{3V_1^2 R_2}{s\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]}$$

Differentiating τ_d with respect to s and setting it equal to zero, it can be shown that the slip for the maximum torque is given by

$$s_{\max,\tau} = \frac{R_2}{\sqrt{R_1^2+(X_1+X_2)^2}}$$

and the corresponding maximum torque (pull-out torque or break-down torque) is given by

$$\tau_{d,\max} = \frac{3V_1^2}{2\omega_s\left[R_1+\sqrt{R_1^2+(X_1+X_2)^2}\right]}$$

Example 6.1: The following test data were obtained on a 460V, 4-pole, 60Hz, △-connected three-phase induction motor:

No-load test: power input=380W, line current = 1.15A at rated voltage.

Blocked-rotor test: power input=15W, line current = 2.1A at the line voltage of 21V.

The friction and windage loss is 21W, and the winding resistance between any two lines is 1.2Ω.

Determine (a) the equivalent circuit parameters of the motor, (b) the starting torque and starting current by using the approximate equivalent circuit, (c) the motor speed, developed torque, and efficiency at s=5%, (d) the maximum torque and its corresponding speed, (e) the maximum developed power and its corresponding speed, and (f) plot the developed torque against the slip.

Solution:

(a) The per–phase resistance of the stator is $R_1 = 1.5R_L = 1.5 \times 1.2 = 1.8\Omega$.

From the blocked-rotor test, $V_{br} = 21V$, $I_{br} = \frac{2.1}{\sqrt{3}} = 1.2A$, $P_{br} = \frac{15}{3} = 5W$. Therefore, the equivalent resistance is

$R_e = \frac{P_{br}}{I_{br}^2} = \frac{5}{1.2^2} = 3.5\Omega$

The rotor resistance is $R_2 = R_e - R_1 = 3.5 - 1.8 = 1.7\Omega$

The equivalent impedance is

$|Z_e| = \frac{V_{br}}{I_{br}} = \frac{21}{1.2} = 17.5\Omega$

The equivalent reactance is

$X_e = \sqrt{|Z_e|^2 - R_e^2} = \sqrt{17.5^2 - 3.5^2} = 17.1\Omega$

From the no-load test, $V_{NL} = 460V$, $I_{NL} = \frac{1.15}{\sqrt{3}} = 0.66A$, $P_{NL} = \frac{380}{3} = 127W$,
$P_c = \frac{380-21}{3} = 120W$. Therefore, the equivalent core resistance is

$R_c = \frac{V_{NL}^2}{P_c} = \frac{460^2}{120} = 1763.3\Omega$

The excitation impedance is

$|Z_\phi| = \frac{V_{NL}}{I_{NL}} = \frac{460}{0.66} = 696.97\Omega$

The magnetization reactance is

$$X_m = \frac{1}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2 - \left(\frac{1}{R_c}\right)^2}} = \frac{1}{\sqrt{\left(\frac{1}{696.97}\right)^2 - \left(\frac{1}{1763.3}\right)^2}} = 758.76\Omega$$

(b)

The synchronous speed is

$\omega_s = \frac{4\pi f}{P} = \frac{4\pi\times 60}{4} = 188.5 rad/s$ or $N_s = \frac{120f}{P} = \frac{120\times 60}{4} = 1800rpm$

The phase input voltage is $\hat{V}_1 = 460\angle 0^\circ V$.

The starting torque is

$$\tau_d\Big|_{s=1} = \frac{3V_1^2 R_2}{s\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]}\Bigg|_{s=1} = \frac{3\times 460^2\times 1.7}{188.5\left((3.5+0)^2+(17.1)^2\right)} = 18.8N\cdot m$$

It follows from the approximate equivalent circuit that the rotor current at the time of starting is

$$\hat{I}_2 = \frac{\hat{V}_1}{R_1+R_2+j(X_1+X_2)+\frac{(1-s)R_2}{s}} = \frac{460\angle 0^\circ}{3.5+j17.1+0} = \frac{460(3.5-j17.1)}{(3.5+j17.1)(3.5-j17.1)} = \frac{460(3.5-j17.1)}{3.5^2+17.1^2} = 5.3 - j25.8A$$

The stator current at the time of starting is

$$\hat{I}_1 = \hat{I}_\phi + \hat{I}_2 = \frac{\hat{V}_1}{R_c} + \frac{\hat{V}_1}{jX_m} + \hat{I}_2 = \hat{V}_1\left(\frac{1}{R_c} + \frac{1}{jX_m}\right) + \hat{I}_2$$

$$= (460\angle 0^\circ)\left(\frac{1}{1763.3} + \frac{1}{j758.76}\right) + (5.3 - j25.8)$$

$$= (460)\left(\frac{1}{1763.3} - j\frac{1}{758.76}\right) + (5.3 - j25.8)$$

$$= 5.56 - j26.41 = \sqrt{5.56^2 + 26.41^2}\angle \tan^{-1}\left(\frac{-26.41}{5.56}\right)\frac{180}{\pi} = 27.0\angle -27.0^\circ A$$

(c)

The motor speed at s=0.05 is $\omega_m = (1-s)\omega_s = (1-0.05)\times 188.5 = 179.1rad/s$ or
$N_m = (1-s)N_s = (1-0.05)\times 1800 = 1710rpm$.

The developed torque is

$$\tau_d|_{s=5\%} = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]}\Bigg|_{s=5\%} = \frac{3\times460^2\times1.7}{0.05\times188.5\left(\left(3.5+\frac{(1-0.05)\times1.7}{0.05}\right)^2+(17.1)^2\right)} = 72.$$
$7N \cdot m$

It follows from the approximate equivalent circuit that the rotor current at s=5% is

$$\hat{I}_2 = \frac{\hat{V}_1}{R_1+R_2+j(X_1+X_2)+\frac{(1-s)R_2}{s}} = \frac{460\angle0^\circ}{3.5+j17.1+\frac{(1-0.05)\times1.7}{0.05}} = \frac{460(35.8-j17.1)}{(35.8+j17.1)(35.8-j17.1)} = \frac{460(35.8-j17.1)}{35.8^2+17.1^2}$$

$$= 10.46 - j5.0 = \sqrt{10.46^2 + 5.0^2}\angle\tan^{-1}\left(\frac{-5.0}{10.46}\right)\frac{180}{\pi} = 11.6\angle-25.5^\circ A$$

The stator current at the time of starting is

$$\hat{I}_1 = \hat{I}_\phi + \hat{I}_2 = \frac{\hat{V}_1}{R_c} + \frac{\hat{V}_1}{jX_m} + \hat{I}_2 = \hat{V}_1\left(\frac{1}{R_c} + \frac{1}{jX_m}\right) + \hat{I}_2$$

$$= (460\angle0^\circ)\left(\frac{1}{1763.3} + \frac{1}{j758.76}\right) + (10.46 - j5.0)$$

$$= (460)\left(\frac{1}{1763.3} - j\frac{1}{758.76}\right) + (10.46 - j5.0)$$

$$= 10.72 - j5.61 = \sqrt{10.72^2 + 5.61^2}\angle\tan^{-1}\left(\frac{-5.61}{10.72}\right)\frac{180}{\pi} = 12.1\angle-27.6^\circ A$$

The input power is

$$P_{in} = 3V_1I_1\cos\theta = 3\times460\times12.1\times\cos\left(27.6\frac{\pi}{180}\right) = 14798W$$

The stator copper loss is $P_{scu} = 3I_2^2R_1 = 3\times11.6^2\times1.8 = 727W($

The air-gap power is $P_{ag} = P_{in} - P_{scu} = 14798 - 727 = 14071W$

The rotor copper loss: $P_{rcu} = 3I_2^2R_2 = 3\times11.6^2\times1.7 = 686W$

The developed power: $P_d = P_{ag} - P_{rcu} = 14071 - 686 = 13385W$

The output power: $P_o = P_d - P_r = P_d - P_c - P_{fw} = 13385 - (380 - 21) - 21 = 13005W$

The efficiency is

$$\eta = \frac{P_o}{P_{in}}\times100 = \frac{13005}{14798}\times100 = 87.9\%$$

(d)

The slip for the maximum torque is

$$s_{\max,\tau} = \frac{R_2}{\sqrt{R_1^2+(X_1+X_2)^2}} = \frac{1.7}{\sqrt{1.8^2+(17.1)^2}} = 0.099$$

and the corresponding maximum torque (pull-out torque or break-down torque) is given by

$$\tau_{d,\max} = \frac{3V_1^2}{2\omega_s\left[R_1+\sqrt{R_1^2+(X_1+X_2)^2}\right]} = \frac{3\times460^2}{2\times188.5\times\left(1.8+\sqrt{1.8^2+(17.1)^2}\right)} = 88.6N\cdot m$$

The speed is

$$N_m = (1-s)N_s = (1-0.099)\times1800 = 1622rpm$$

(e)

The slip for the maximum developed power is

$$s_{\max,p} = \frac{R_2}{R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}} = \frac{1.7}{1.7+\sqrt{3.5^2+17.1^2}} = 0.089$$

and

$$P_{d,\max} = \frac{3}{2}\frac{V_1^2}{R_1+R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}} = \frac{3}{2}\frac{460^2}{3.5+\sqrt{3.5^2+17.1^2}} = 15147W$$

The speed is

$$N_m = (1-s)N_s = (1-0.089)\times 1800 = 1640rpm$$

or

$$\omega_m = (1-s)\omega_s = (1-0.089)\times 188.5 = 171.7rad/s$$

The developed torque is

$$\tau_d = \frac{P_{d,\max}}{\omega_m} = \frac{15147}{171.7} = 88.2N\cdot m$$

(f) The relationship between the developed torque and the slip is given by

$$\tau_d = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]} = \frac{3\times 460^2\times 1.7}{s\times 188.5\times\left(\left(3.5+\frac{(1-s)\times 1.7}{s}\right)^2+(17.1)^2\right)} = \frac{5725.0}{s\left(\left(\frac{1}{s}(1.7s-1.7)-3.5\right)^2+292.41\right)}$$

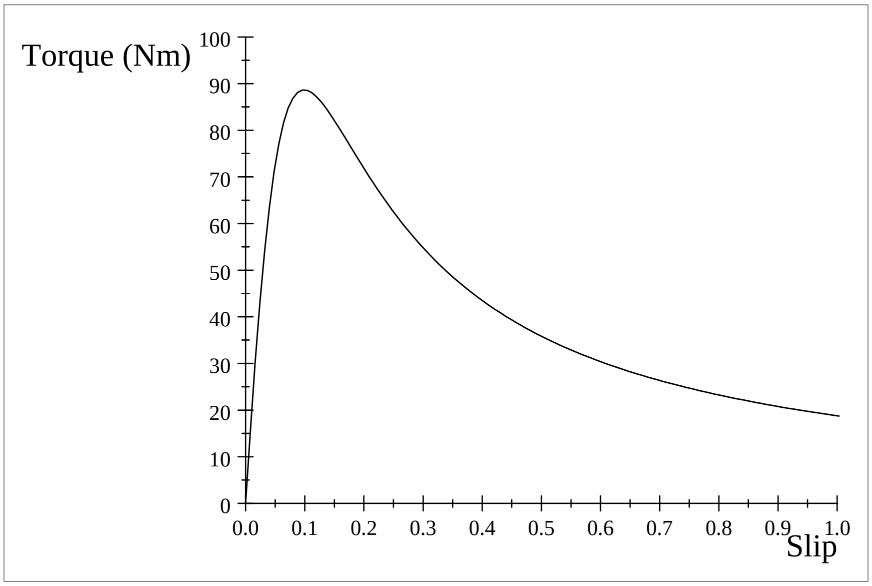


Figure 6.6

Some Important Observation:

(1) When the motor is operating near its rated slip, which is less than 10%, the developed torque is directly proportional slip.

(2) For a constant slip, the developed torque is directly proportional the square of the applied voltage.

6.2 Single-Phase Induction Motors

6.2.1 Double Revolving-Field Theory

Similar to three-phase induction motors, single-phase motors have the stator and rotor with the rotor winding short-circuited. Unlike three-phase induction motors, single-phase induction motors have only one phase winding in the stator. A single-phase AC voltage is applied to the stator winding, which produces an AC current in the stator winding. Suppose that $i(t) = I_m \cos(\omega t)$. Then, the resultant air-gap magnetic flux density is given by

$$\vec{B} = B_{\max}\cos(\omega t)\vec{i} = \vec{B}_{CW} + \vec{B}_{CCW}$$

where $\vec{B}_{CW}$ and $\vec{B}_{CCW}$ represent the clockwise and counterclockwise rotating magnetic fields, respectively, defined by

$$\vec{B}_{CW} = 0.5B_m\cos(\omega t)\vec{i} - 0.5B_m\sin(\omega t)\vec{j}$$

$$\vec{B}_{CCW} = 0.5B_m\cos(\omega t)\vec{i} + 0.5B_m\sin(\omega t)\vec{j}$$

The equation above implies that the sum of the clockwise and counterclockwise rotating magnetic fields is equal to the stationary pulsating megnetic field, that is, the stationary pulsating megnetic field can be resolved into two rotating magnetic fields, each of equal magnetude but rotating at the synchronous speed in opposite directions. The synchronous speed is determined by

$$\omega_s = \frac{4\pi f}{P} \text{ or } N_s = \frac{120f}{P}$$

Suppose that the motor rotates in the counterclockwise direction at speed ω_m. Define $s = \frac{\omega_s - \omega_m}{\omega_s}$. Then, $\vec{B}_{CCW}$ is called the forward magnetic field, denoted $\vec{B}_f$, rotating at the synchronous speed of $\omega_{fs} = \omega_s$ while $\vec{B}_{CW}$ the backward magnetic field at $\omega_{bs} = -\omega_s$, denoted $\vec{B}_b$. The slip of the motor is $s_f = \frac{\omega_{fs} - \omega_m}{\omega_{fs}} = s$ with respect to $\vec{B}_f$ and $s_b = \frac{\omega_{rs} - \omega_m}{\omega_{rs}} = \frac{-\omega_s - \omega_m}{-\omega_s} = \frac{-2\omega_s + \omega_s - \omega_m}{-\omega_s} = 2 - s$ with respect to $\vec{B}_b$.

Similar to three-phase induction motors, single-phase motors can be analyzed by using the equivalent circuit. Figure 6.7 shows the equivalent circuit for a single-phase AC motor at still, which is equivalent to the circuit with the effects of the forward and backward magnetic fields separated, as shown in Figure 6.8. For a motor running at speed ω_m, the effective rotor resistance changes with the slip. The rotor resistance is $\frac{R_2}{s}$ with respect to $\vec{B}_f$ while $\frac{R_2}{2-s}$ with respect to $\vec{B}_b$. The final equivalent circuit is shown in Figure 6.9.

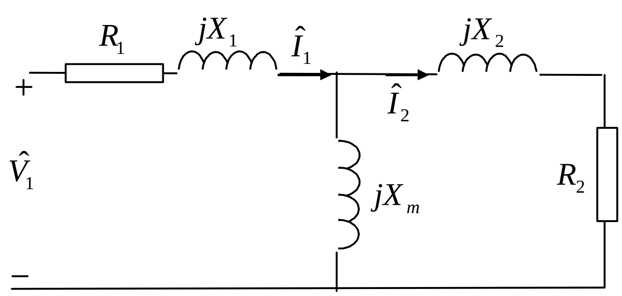


Figure 6.7

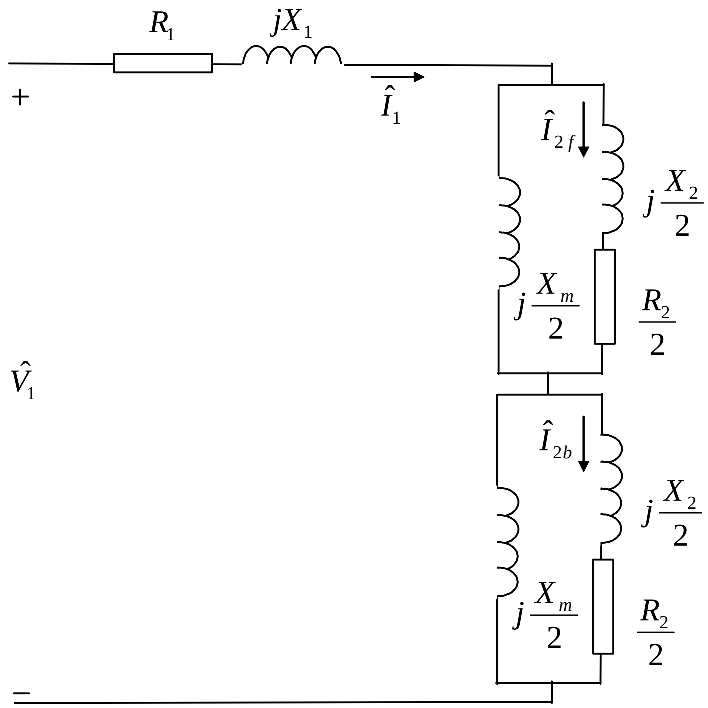


Figure 6.8

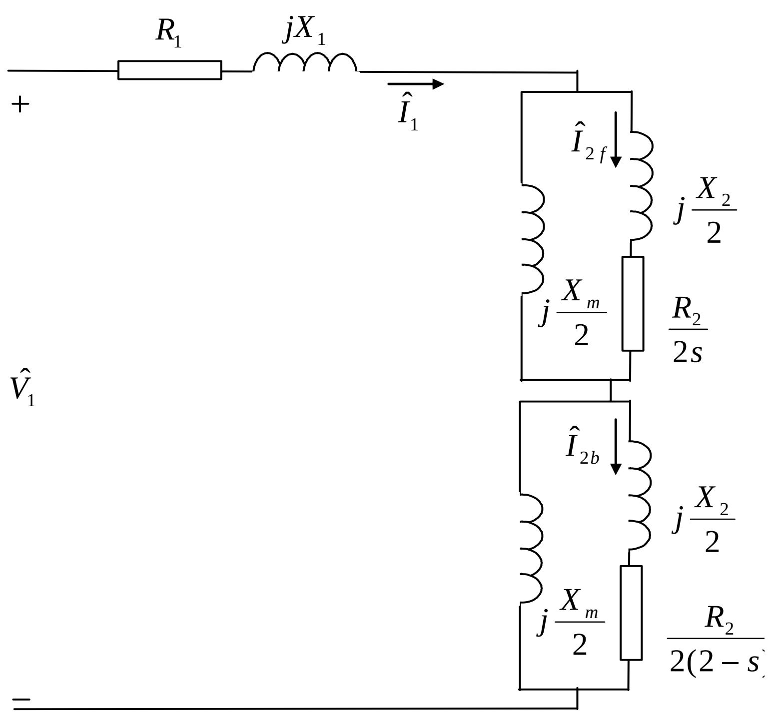


Figure 6.9

Define

$$Z_1 = R_1 + jX_1$$

$$Z_f = R_f + jX_f = 0.5\frac{jX_m(R_2/s+jX_2)}{R_2/s+j(X_2+X_m)} = 0.5\frac{jX_m(R_2/s+jX_2)((R_2/s-j(X_2+X_m)))}{(R_2/s+j(X_2+X_m))(R_2/s-j(X_2+X_m))}$$
$$= 0.5\frac{jX_m\left((R_2/s)^2+X_2(X_2+X_m)-jX_m(R_2/s)\right)}{(R_2/s)^2+(X_2+X_m)^2} = 0.5\frac{X_m^2(R_2/s)+jX_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$$
$$= 0.5\frac{X_m^2(R_2/s)}{(R_2/s)^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$$
$$Z_b = R_b + jX_b = 0.5\frac{jX_m(R_2/(2-s)+jX_2)}{R_2/(2-s)+j(X_2+X_m)} = 0.5\frac{jX_m(R_2/(2-s)+jX_2)((R_2/(2-s)-j(X_2+X_m)))}{(R_2/(2-s)+j(X_2+X_m))(R_2/(2-s)-j(X_2+X_m))}$$
$$= 0.5\frac{jX_m\left((R_2/(2-s))^2+X_2(X_2+X_m)-jX_m(R_2/(2-s))\right)}{(R_2/(2-s))^2+(X_2+X_m)^2} = 0.5\frac{X_m^2(R_2/(2-s))+jX_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$$
$$= 0.5\frac{X_m^2(R_2/(2-s))}{(R_2/(2-s))^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$$
$$R_f = 0.5\frac{X_m^2(R_2/s)}{(R_2/s)^2+(X_2+X_m)^2}$$
$$X_f = 0.5\frac{X_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$$
$$R_b = 0.5\frac{X_m^2(R_2/(2-s))}{(R_2/(2-s))^2+(X_2+X_m)^2}$$
$$X_b = 0.5\frac{X_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$$

Then,
$$Z_{in} = Z_1 + Z_f + Z_b$$

Thus, the stator current is
$$\hat{I}_1 = \frac{\hat{V}_1}{Z_{in}}$$

The input power is
$$P_{in} = \mathrm{Re}\left[\hat{V}_1 \hat{I}_1^*\right] = V_1 I_1 \cos\theta$$

The stator copper loss is
$$P_{scu} = I_1^2 R_1$$

The air-gap power due to the forward magnetic field is
$$P_{agf} = I_1^2 R_f = 0.5 I_{2f}^2 \frac{R_2}{s}$$

The air-gap power due to the backward magnetic field is
$$P_{agb} = I_1^2 R_b = 0.5 I_{2b}^2 \frac{R_2}{2-s}$$

The forward rotor copper loss is
$$P_{rcuf} = 0.5 I_{2f}^2 R_2 = sP_{agf}$$

The backward rotor copper loss is
$$P_{rcub} = 0.5 I_{2b}^2 R_2 = (2-s)P_{agb}$$

The power developed by the forward magnetic field is
$$P_{df} = P_{agf} - P_{rcuf} = (1-s)P_{agf}$$

The power developed by the backward magnetic field is
$$P_{db} = P_{agb} - P_{rcub} = -(1-s)P_{agb}$$

The total developed power is
$$P_d = P_{df} + P_{db} = (1-s)(P_{agf} - P_{agb}) = (1-s)P_{ag}$$

So, the net air-gap power is

$$P_{ag} = P_{agf} - P_{agb}$$

The mechanical developed power is

$$P_d = (1-s)P_{ag} = \tau_d \omega_m = (1-s)\tau_d \omega_s$$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_m} = \frac{(1-s)P_{ag}}{(1-s)\omega_s} = \frac{P_{agf}-P_{agb}}{\omega_s} = \frac{P_{agf}}{\omega_s} - \frac{P_{agb}}{\omega_s} = \tau_{fd} - \tau_{bd}$$

The output power is

$$P_o = P_d - P_r$$

where the rotational loss is $P_r = P_c + P_{fw} + P_{stray}$.

Example 6.2: A 4-pole 110V 50Hz single-phase induction motor has $R_1 = 2\Omega$, $X_1 = 2.8\Omega$, $R_2 = 3.8\Omega$, $X_2 = 2.8\Omega$, and $X_m = 60\Omega$. The rotational loss is 20W. Determine the shaft torque, the motor efficiency when the slip is 4%, and the developed torque characteristics.

Solution:The synchronous speed is

$$\omega_s = \frac{4\pi f}{P} = \frac{4\pi 50}{4} = 157.08 rad/s \text{ or } N_s = \frac{120f}{P} = \frac{120\times 50}{4} = 1500 rpm$$

The impedances are

$$Z_1 = R_1 + jX_1 = 2 + j2.8$$

$$Z_f = R_f + jX_f = \frac{0.5jX_m(0.5R_2/s+j0.5X_2)}{0.5jX_m+(0.5R_2/s+j0.5X_2)} = 0.5\frac{jX_m(R_2/s+jX_2)}{R_2/s+j(X_2+X_m)} = 0.5\frac{j60(3.8/0.04+j2.8)}{3.8/0.04+j(2.8+60)}$$

$$= 0.5\frac{-168+j5700}{95.0+j62.8} = 0.5\frac{(-168+j5700)(95-j62.8)}{(95+j62.8)(95-j62.8)} = 0.5\frac{3.42\times 10^5+j5.5205\times 10^5}{95^2+62.8^2} = 13.185 + j21.284\Omega$$

$$Z_b = R_b + jX_b = \frac{0.5jX_m[0.5R_2/(2-s)+j0.5X_2]}{0.5jX_m+[0.5R_2/(2-s)+j0.5X_2]} = 0.5\frac{jX_m[R_2/(2-s)+jX_2]}{R_2/(2-s)+j(X_2+X_m)} = 0.5\frac{j60\times(3.8/(2-0.04)+j2.8)}{3.8/(2-0.04)+j(2.8+60)}$$

$$= 0.5\frac{-168.0+j116.33}{1.9388+j62.8} = 0.5\frac{(-168.0+j116.33)(1.9388-j62.8)}{(1.9388+j62.8)(1.9388-j62.8)} = \frac{0.5(6979.8+j10776)}{1.9388^2+62.8^2} = 0.88406 + j1.3649\Omega$$

$$Z_{in} = Z_1 + Z_f + Z_b = (2+j2.8) + (13.185+j21.284) + (0.88406+j1.3649) = 16.069 + j25.449\Omega$$

Thus, the stator current is

$$\hat{I}_1 = \frac{\hat{V}_1}{Z_{in}} = \frac{110\angle 0^\circ}{16.069+j25.449} = \frac{110\angle 0^\circ}{\sqrt{16.069^2+25.449^2}\angle \tan^{-1}\left(\frac{25.449}{16.069}\right)\frac{180}{\pi}} = \frac{110\angle 0^\circ}{30.098\angle 57.731^\circ} = 3.6547\angle -57.731^\circ A$$

$$\hat{I}_{2f} = \frac{j\frac{X_m}{2}}{\frac{R_2}{2s}+j\left(\frac{X_2}{2}+\frac{X_m}{2}\right)}\hat{I}_1 = \frac{jX_m}{\frac{R_2}{s}+j(X_2+X_m)}\hat{I}_1 = \frac{j60}{\frac{3.8}{0.04}+j(2.8+60)}(3.6547\angle -57.731^\circ)$$

$$= \frac{(60\angle 90^\circ)(3.6547\angle -57.731^\circ)}{\sqrt{\left(\frac{3.8}{0.04}\right)^2+(2.8+60)^2}\angle \tan^{-1}\left(\frac{2.8+60}{\frac{3.8}{0.04}}\right)\frac{180}{\pi}} = \frac{(60\angle 90^\circ)(3.6547\angle -57.731^\circ)}{113.88\angle 33.467^\circ} = 1.9256\angle -1.198^\circ A$$

$$\hat{I}_{2b} = \frac{j\frac{X_m}{2}}{\frac{R_2}{2(2-s)}+j\left(\frac{X_2}{2}+\frac{X_m}{2}\right)}\hat{I}_1 = \frac{jX_m}{\frac{R_2}{s}+j(X_2+X_m)}\hat{I}_1 = \frac{j60}{\frac{3.8}{2-0.04}+j(2.8+60)}(3.6547\angle -57.731^\circ)$$

$$= \frac{(60\angle 90^\circ)(3.6547\angle -57.731^\circ)}{\sqrt{\left(\frac{3.8}{2-0.04}\right)^2+(2.8+60)^2}\angle \tan^{-1}\left(\frac{2.8+60}{\frac{3.8}{2-0.04}}\right)\frac{180}{\pi}} = \frac{(60\angle 90^\circ)(3.6547\angle -57.731^\circ)}{62.830\angle 88.232^\circ} = 3.4901\angle -55.963^\circ A$$

The input power is

$$P_{in} = \mathrm{Re}\left[\hat{V}_1 \hat{I}_1^*\right] = V_1 I_1 \cos\theta = 110 \times 3.6547 \times \cos\left(57.731\frac{\pi}{180}\right) = 214.63W$$

The stator copper loss is

$$P_{scu} = I_1^2 R_1 = 3.6547^2 \times 2 = 26.714W$$

The air-gap power due to the forward magnetic field is

$P_{agf} = I_1^2 R_f = 3.6547^2 \times 13.185 = 176.11W$ or $= 0.5 I_{2f}^2 \frac{R_2}{s} = 0.5 \times 1.9256^2 \frac{3.8}{0.04} = 176.13W$

The air-gap power due to the backward magnetic field is

$P_{agb} = I_1^2 R_b = 3.6547^2 \times 0.88406 = 11.808W$ or $= 0.5 I_{2b}^2 \frac{R_2}{2-s} = 0.5 \times 3.4901^2 \times \frac{3.8}{2-0.04} = 11.808W$

The net air-gap power is

$$P_{ag} = P_{agf} - P_{agb} = 176.13 - 11.808 = 164.32W$$

The mechanical developed power is

$$P_d = (1-s)P_{ag} = (1-0.04) \times 164.32 = 157.75W$$

The output power is

$$P_o = P_d - P_r = 157.75 - 20 = 137.75W$$

The efficiency is

$$\eta = \frac{137.75}{214.63} \times 100 = 64.2\%$$

The motor speed is

$$\omega_m = (1-s)\omega_s = (1-0.04) \times 157.08 = 150.80 rad/s$$

The motor shaft torque is

$$\tau_o = \frac{P_o}{\omega_m} = \frac{137.75}{150.80} = 0.91346 N \cdot m$$

The developed torque of the forward and backward magnetic field is

$$\tau_{df} = \frac{(1-s)P_{agf}}{\omega_m} = \frac{V_1^2 R_f}{\omega_s\left((R_1+R_f+R_b)^2+(X_1+X_f+X_b)^2\right)} = \frac{110^2 R_f}{157.08\left((2+R_f+R_b)^2+(2.8+X_f+X_b)^2\right)}$$

$$\tau_{db} = \frac{V_1^2 R_b}{\omega_s\left((R_1+R_f+R_b)^2+(X_1+X_f+X_b)^2\right)} = \frac{110^2 R_b}{157.08\left((2+R_f+R_b)^2+(2.8+X_f+X_b)^2\right)}$$

with

$$R_f = 0.5\frac{60^2(3.8/s)}{(3.8/s)^2+(2.8+60)^2}$$

$$X_f = 0.5\frac{60\left((3.8/s)^2+2.8(2.8+60)\right)}{(3.8/s)^2+(2.8+60)^2}$$

$$R_b = 0.5\frac{60^2(3.8/(2-s))}{(3.8/(2-s))^2+(2.8+60)^2}$$

$$X_b = 0.5\frac{60\left((3.8/(2-s))^2+2.8(2.8+60)\right)}{(3.8/(2-s))^2+(2.8+60)^2}$$

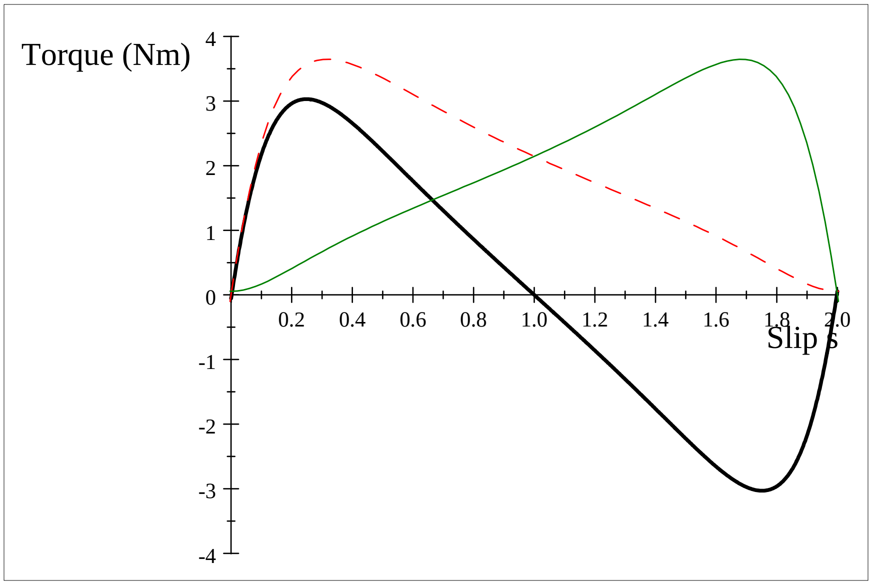


Figure 6.10 τ_d (thick solid line) τ_{df} (dashed line); τ_{df} (thin

$$\tau_d = \frac{110^2 R_f}{157.08\left((2+R_f+R_b)^2+(2.8+X_f+X_b)^2\right)} - \frac{110^2 R_b}{157.08\left((2+R_f+R_b)^2+(2.8+X_f+X_b)^2\right)}$$

6.2.2 Types of Single-Phase Induction Motors

It is observed from Figure 6.10 that the developed torque is the torque developed by the forward magnetic field less the torque developed by the backward magnetic field. It is noted that τ_{df} and τ_{db} are the same at the starting moment, so the starting torque is zero, which means that this motor cannot start by itself. However, by introducing an extra winding and some capacitors, single-phase induction motors can be made self-starting.

1. Split-Phase Motors

A split-phase induction motor has two separate windings: main winding and auxiliary winding. They are placed in space quadrature and connected to a single-phase power source. The main winding has a low resistance and high inductance and carries current to establish the main flux at the rated speed. The auxiliary winding has a high resistance and low inductance and is disconnected from the supply by a centrifugal switch when the motor reaches a speed of nearly 75% of its synchronous speed.

At the time of starting, the main winding current lags the applied voltage by almost 90° owing to its high inductance (large number of turns) and low resistance (large size wire) while the auxiliary winding current is essentially in phase with the applied voltage due to its low inductance and high resistance. Since the two windings are placed in space quadrature and carry out-of-phase currents, a rotating magnetic field is produced in the air-gap and the motor is able to rotate by itself.

2. Capacitor-Start Motors

In split-phase motor, the main winding current does not lag the auxiliary winding current exactly by 90°. However, by connecting a capacitor in series with the auxiliary winding, it is possible to make the main winding current lag the auxiliary winding current exactly by 90°.

3. Capacitor-Start Capacitor Run Motors

The power factor for both split-phase and capacitor start motors is low and so is efficiency, usually 50%-60%. The efficiency can be improved by employing another capacitor when the motor runs at the rated speed. This led to the development of a capacitor-start and capacitor-run motor.

4. Permanent Split-Capacitor Motors

The permanent split-capacitor motor is developed by removing the start-capacitor and cetrifugal switch from the capacitor-start capacitor-run motor.

Chapter 7. Special Motors

7.1 Universal Motors

A DC series motor specially designed for AC operation is usually referred to as a universal motor. The equivalent circuit is shown in Figure 7.1

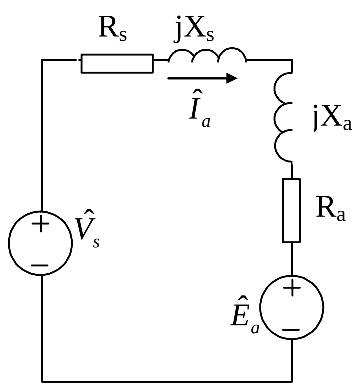


Figure 7.1

The phasor diagram for a lagging load is shown in Figure 7.2.

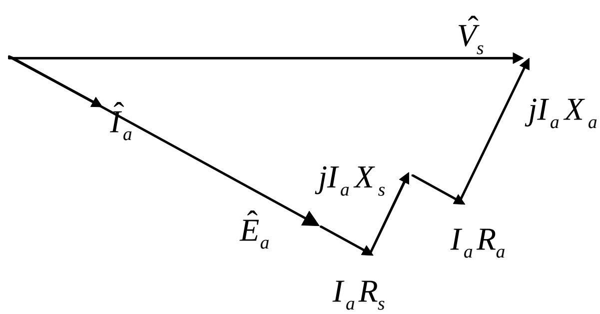


Figure 7.2

Example 7.1: A 120V 60Hz 2-pole universal motor operates at a speed of 8000rpm on full load and draws a current of 17.58A at a lagging power factor of 0.912. The impedance of the series field winding is 0.65+j1.2Ω. The impedance of the armature winding is 1.36+j1.6Ω. Determine (a) the induced voltage, (b) the power output, (c) the shaft torque, and (d) the efficiency if the rotational loss is 80W.

Solution: From the equivalent circuit, we have

$$\widehat{E}_a = \widehat{V}_s - \widehat{I}_a(R_s + R_a + jX_s + jX_a) = 120 - (17.58\angle - 24.22^\circ)(0.65 + 1.36 + j(1.2 + 1.6$$
$$= 120 - 17.58\left(\cos\left(-24.22\frac{\pi}{180}\right) + j\sin\left(-24.22\frac{\pi}{180}\right)\right)(0.65 + 1.36 + j(1.2 + 1.6))$$
$$= 67.581 - j30.395 = \sqrt{67.581^2 + 30.395^2}\angle\tan^{-1}\left(\frac{-30.395}{67.581}\right)\frac{180}{\pi} = 74.1\angle - 24.$$
$22^\circ V$

Note that the induced voltage is in phase with the armature current.

The input power is

$$P_{in} = V_s I_a \cos\theta = 120 \times 17.58 \times 0.912 = 1924W$$

The copper loss

$$P_{cu} = I_a^2(R_s + R_a) = 17.58^2(0.65 + 1.36) = 621.2W$$

The developed power is

$$P_d = P_{in} - P_{cu} = 1924 - 621.2 = 1302.8W$$

The output power is

$$P_o = P_d - P_r = 1302.8 - 80 = 1222.8W$$

The efficiency is

$$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{1222.8}{1924} \times 100 = 63.6\%$$

The motor speed is

$$\omega_m = \frac{2\pi n}{60} = \frac{2\pi \times 8000}{60} = 837.76 rad/s$$

The shaft torque is

$$\tau_o = \frac{P_o}{\omega_m} = \frac{1222.8}{837.76} = 1.46 N \bullet m$$

7.2 Permanent DC Motors

A DC motor with the magnetic field being produced by permanent magnets is called the permanent DC motor. The equivalent circuit for a permanent DC motor is shown in Figure 7.3.

The dynamical equations are given by

$$e_a(t) = K_a\Phi_a\omega(t)$$
$$\tau_d(t) = K_a\Phi_a i_a(t)$$
$$v_a(t) = R_a i_a(t) + L_a\frac{di_a(t)}{dt} + K_a\Phi_a\omega(t)$$
$$J\frac{d\omega(t)}{dt} = K_a\Phi_a i_a(t) - \tau_L(t) - D\omega(t)$$

The steady-state quantities are calculated by letting $\frac{di_a(t)}{dt}$ and $\frac{d\omega(t)}{dt}$ be zero, that is,

$$e_a(\infty) = K_a\Phi_a\omega(\infty)$$
$$\tau_d(\infty) = K_a\Phi_a i_a(\infty)$$
$$v_a(\infty) = R_a i_a(\infty) + K_a\Phi_a\omega(\infty)$$
$$0 = K_a\Phi_a i_a(\infty) - \tau_L(\infty) - D\omega(\infty)$$

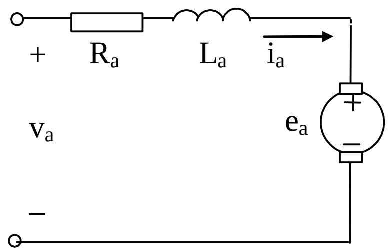


Figure 7.3

Example 7.2: Calculate the magnetic flux in a 200W, 100V PM DC motor operating at 1500rpm. The motor constant is 85, the armature resistnace is 2Ω, and the rotational loss is 15W.

Solution: $\omega(\infty) = \frac{2\pi n}{60} = \frac{2\pi\times1500}{60} = 157.08 rad/s$

The developed power is $P_d = P_o + P_r = 200 + 15 = 215W$

The developed torque is $\tau_d(\infty) = \frac{P_d}{\omega(\infty)} = \frac{215}{157.08} = 1.3687 N\cdot m$

It follows from $\tau_d(\infty) = K_a\Phi_a i_a(\infty)$ that

$$i_a(\infty) = \frac{\tau_d(\infty)}{K_a\Phi_a}$$

Substituting this into $v_a(\infty) = R_a i_a(\infty) + K_a\Phi_a\omega(\infty)$ gives

$$v_a(\infty) = R_a\frac{\tau_d(\infty)}{K_a\Phi_a} + K_a\Phi_a\omega(\infty)$$

that is,

$$100 = 2\frac{1.3687}{85\Phi_a} + 85\times157.08\Phi_a^2$$

or

$$8500\Phi_a = 2\times1.3687 + 85^2\times157.08\Phi_a^2$$

Solving this for positive Φ_a produces

$$\Phi_a = \frac{8500\pm\sqrt{8500^2-4\times2\times1.3687\times85^2\times157.08}}{2\times85^2\times157.08} = 7.1524\times10^{-3} \text{ and } 3.3723\times10^{-4}$$

Because $e_a(\infty) = K_a\Phi_a\omega(\infty) = 85\times3.3723\times10^{-4}\times157.08 = 4.5026V$ is too small and $e_a(\infty) = K_a\Phi_a\omega(\infty) = 85\times7.1524\times10^{-3}\times157.08 = 95.497V$ is reasonable, so $\Phi_a = 7.1524\times10^{-3}Wb$.

7.2 Stepper Motors

$$\theta_m = \frac{2}{P}\theta_e$$

$$\omega_m = \frac{2}{P}\omega_e$$

$$n_m = \frac{2}{P}n_e$$

$$n_e = \frac{1}{2N}n_{pulses}$$

$$n_m = \frac{1}{NP}n_{pulses}$$

where P is the number of poles, N is the number of phases, θ_m is the mechanical angle, θ_e is the electrical angle, ω_m and n_m are the mechanical speed, ω_e and n_e are the electrical speed, n_{pulses} is the number of pules per minute.

Example 7.3: A three-phase permanent-magnet stepper motor required for one particular application must be capable of controlling the position of a shaft in steps of 7.5°, and it nust be capable of running at speeds of up to 300rpm. (a) How many poles must this motor have? (b) At what rate must control pulses be received in the motor's control unit if it is to be driven at 300rpm?

Solution: (a) In a three-phase stepper motor, each pulse advances the rotor's position by 60 electrical degrees. This advance must correspond to 7.5 mechanical degrees. Solving $\theta_m = \frac{2}{P}\theta_e$ for P yields

$$P = 2\frac{\theta_e}{\theta_m} = 2\frac{60}{7.5} = 16 \text{ poles}$$

(b) Solving $n_m = \frac{1}{NP}n_{pulses}$ for n_{pulses} gives

$$n_{pulses} = NPn_m = 3 \times 16 \times 300 = 14400 \text{ pulses/minute=240 pulses/s}$$

Formula Sheet for the Final Exam:

$\vec{B} = \mu\vec{H}, \phi = BA, \mu_0 = 4\pi \times 10^{-7} H/m$

$\phi = \frac{\mathcal{F}}{\Re}, \mathcal{F} = Ni, \Re = \frac{l}{\mu A}$

$e = \frac{d\lambda}{dt}, \lambda = N\phi, L = \frac{\lambda}{i} = \frac{N^2}{\Re}$

$W_\phi(\lambda, x) = \frac{1}{2L}\lambda^2, W_\phi(i, x) = \frac{1}{2}Li^2$

$f = -\frac{\partial W_\phi(\lambda,x)}{\partial x} = \frac{1}{2}i^2\frac{dL(x)}{dx}, f = \frac{\partial W_\phi(i,x)}{\partial x} = \frac{1}{2}i^2\frac{dL(x)}{dx}$

$e_1 = \frac{d\lambda_1}{dt}, e_2 = \frac{d\lambda_2}{dt}, \lambda_1 = \lambda_{11} + \lambda_{12} = L_{11}i_1 + L_{12}i_2, \lambda_2 = \lambda_{21} + \lambda_{22} = L_{21}i_1 + L_{22}i_2$

$W_\phi(\lambda_1, \lambda_2, \theta) = \frac{1}{2}\Gamma_{11}^2\lambda_1^2 + \Gamma_{12}\lambda_1\lambda_2 + \frac{1}{2}\Gamma_{22}\lambda_2^2, W_\phi(i_1, i_2, \theta) = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1i_2 + \frac{1}{2}L_{22}i_2^2$

$\tau = -\frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial\theta} = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$

$\tau = \frac{\partial W_\phi(i_1,i_2,\theta)}{\partial\theta} = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$

$e = l\vec{v} \times \vec{B}, f = i\vec{l} \times \vec{B}$

$e_a(t) = K_e i_f(t)\omega(t), \tau_d(t) = K_\tau i_f(t) i_a(t)$

$\omega(s) = \frac{K_e I_f(V_a(s)+L_a i_a(0))+(L_a s+R_a)(J\omega(0)-\tau_L(s))}{(Js+D)(L_a s+R_a)+(K_e I_f)^2}$

$I_a(s) = \frac{(V_a(s)+L_a i_a(0))(Js+D)-K_e I_f(J\omega(0)-\tau_L(s))}{(Js+D)(L_a s+R_a)+(K_e I_f)^2}$

$I_f(s) = \frac{V_f(s)+L_f i_f(0)}{L_f s+R_f}$

$Z_Y = \frac{1}{3}Z_\triangle, \hat{V}_{an} = \frac{\hat{V}_{ab}}{\sqrt{3}}\angle -30^\circ, \hat{I}_{A'} = \sqrt{3}\,\hat{I}_A\angle -30^\circ$

$\triangle$-**Y connection**: $\hat{E}_{A_1} = a\hat{E}_{A_2}\angle -30^\circ$, $\hat{I}'_{A_2} = \frac{1}{a}\hat{I}_{A_2}\angle -30^\circ$

Y-$\triangle$ connection: $\hat{E}_{A_1} = a\hat{E}_{A_2}\angle 30^\circ$, $\hat{I}'_{A_2} = \frac{1}{a}\hat{I}_{A_2}\angle 30^\circ$

$\omega_m = (1-s)\omega_s, \omega_s = \frac{4\pi f}{P}$

$\hat{E}'_a = \hat{E}_a - j\hat{I}_d(X_d - X_q)$(synchronous generator),$\hat{E}'_a = \hat{E}_a + j\hat{I}_d(X_d - X_q)$(synchronous motor)

$P_d = 3I_2^2\frac{(1-s)R_2}{s} = \frac{3V_1^2\frac{(1-s)R_2}{s}}{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}, \tau_d = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+\frac{R_2}{s}\right)^2+(X_1+X_2)^2\right]}$

$s_{\max,p} = \frac{R_2}{R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}, P_{d,\max} = \frac{3}{2}\frac{V_1^2}{R_1+R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}$

$s_{\max,\tau} = \frac{R_2}{\sqrt{R_1^2+(X_1+X_2)^2}}, \tau_{d,\max} = \frac{3V_1^2}{2\omega_s\left[R_1+\sqrt{R_1^2+(X_1+X_2)^2}\right]}$

$Z_f = R_f + jX_f = 0.5\frac{jX_m(R_2/s+jX_2)}{R_2/s+j(X_2+X_m)} = 0.5\frac{X_m^2(R_2/s)}{(R_2/s)^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$

$Z_b = R_b + jX_b = 0.5\frac{jX_m(R_2/(2-s)+jX_2)}{R_2/(2-s)+j(X_2+X_m)} = 0.5\frac{X_m^2(R_2/(2-s))}{(R_2/(2-s))^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$

$P_{agf} = I_1^2R_f = 0.5I_{2f}^2\frac{R_2}{s}, P_{agb} = I_1^2R_b = 0.5I_{2b}^2\frac{R_2}{2-s}$

$P_{df} = P_{agf} - P_{rcuf} = (1-s)P_{agf}, P_{db} = P_{agb} - P_{rcub} = -(1-s)P_{agb}$

$P_d = (1-s)P_{ag}, P_{ag} = P_{agf} - P_{agb}$

$P_d = (1-s)P_{ag} = \tau_d\omega_m = (1-s)\tau_d\omega_s$

$$\tau_d = \frac{P_{agf}}{\omega_s} - \frac{P_{agb}}{\omega_s} = \tau_{fd} - \tau_{bd}$$
$$e_a(t) = K_a\Phi_a\omega(t), \tau_d(t) = K_a\Phi_a i_a(t)$$
$$\theta_m = \frac{2}{P}\theta_e, n_m = \frac{1}{NP}n_{pulses}$$