Chapter 4 Transformers

4.1 Ideal Transformers (Lecture 12)

Figure 4.1 shows a transformer circuit.

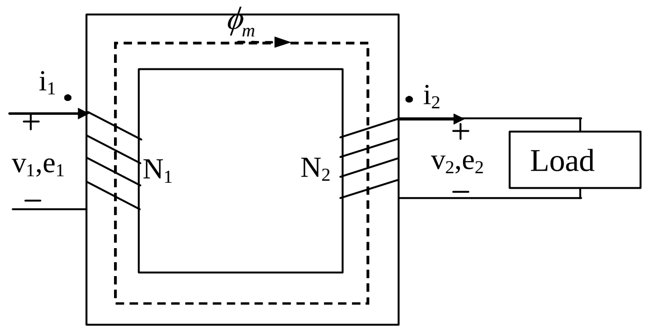


Figure 4.1 An ideal transformer

The dot markings indicate terminals of corresponding polarity, that is, if one follows through the primary and secondary windings beginning at their marked terminals, one will find that both windings encircle the core in the same direction with respect to the flux. Therefore, if one compares the voltages of the two windings, the voltages from the dot-marked to an unmarked terminal will have the same instantaneous polarity for both windings.

A transformer is called the ideal transformer if the following assumptions are satisfied:

(A1) The core of the transformer is highly permeable so that is requires vanishingly small magnetomotive force (mmf) to set up the flux ϕ.

(A2) There is no eddy-current or hysteresis loss.

(A3) There is no resistance.

(A4) There is no leakage flux.

With these assumptions, it is obvious that

$$v_1 = e_1 = \frac{d\lambda_1}{dt} = \frac{d(N_1\phi)}{dt} = N_1\frac{d\phi}{dt}$$
$$v_2 = e_2 = \frac{d\lambda_2}{dt} = \frac{d(N_2\phi)}{dt} = N_2\frac{d\phi}{dt}$$

which implies that

$$\frac{v_1}{v_2} = \frac{e_1}{e_2} = \frac{N_1}{N_2} = a$$

where a is referred to as turns ratio or transformation ratio.

Since there is no loss in the ideal transformer, the input power is the same as the output power, that is,

$$v_1 i_1 = v_2 i_2$$

As a result, we have

$$\frac{i_2}{i_1} = \frac{v_1}{v_2} = a$$

Now suppose that the instantaneous flux is $\phi = \phi_{\max}\sin(\omega t)$. Then we have

$$v_1 = e_1 = N_1\frac{d\phi}{dt} = N_1\omega\phi_{\max}\cos(\omega t) = \sqrt{2}\,V_1\cos(\omega t)$$
$$v_2 = e_2 = N_2\frac{d\phi}{dt} = N_2\omega\phi_{\max}\cos(\omega t) = \sqrt{2}\,V_2\cos(\omega t)$$

where $V_1 = \frac{N_1 \omega \phi_{\max}}{\sqrt{2}}$ and $V_2 = \frac{N_2 \omega \phi_{\max}}{\sqrt{2}}$ are rms values of v_1 and v_2. Note that there is no phase shift between e_1 and e_2.

It is a common practice to express sinusoidal signals $i_1, i_2, e_1, e_2, v_1,$ and v_2 in terms of phasors as $\hat{I}_1, \hat{I}_2, \hat{E}_1, \hat{E}_2, \hat{V}_1,$ and $\hat{V}_2$. Then, we have

$$\frac{\hat{I}_2}{\hat{I}_1} = \frac{\hat{E}_1}{\hat{E}_2} = \frac{\hat{V}_1}{\hat{V}_2} = a$$

If $\hat{Z}_2$ is the load impedance on the secondary side, then

$$\hat{Z}_2 = \frac{\hat{V}_2}{\hat{I}_2} = \frac{\frac{\hat{V}_1}{a}}{a\hat{I}_1} = \frac{1}{a^2}\frac{\hat{V}_1}{\hat{I}_1} = \frac{1}{a^2}\hat{Z}_1$$

where $\hat{Z}_1 = \frac{\hat{V}_1}{\hat{I}_1}$ is the load impedance as referred to the primary side. The equivalent circuit for an ideal transformer is shown in Figure 4.2.

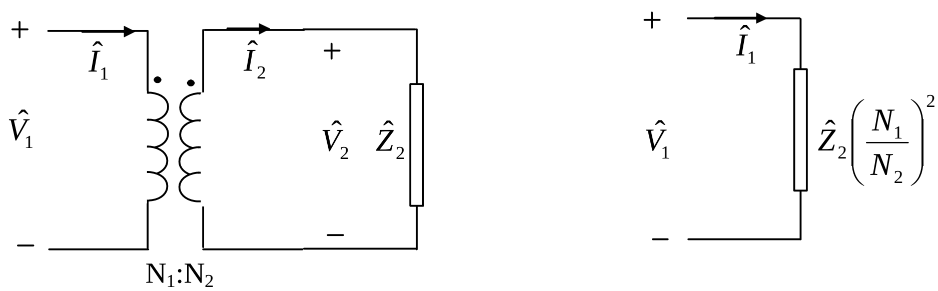


Figure 4.2 The equivalent circuit of an ideal transformer

4.2 Practical Transformers

For a practical transformer, both primary and secondary windings have resistances, denoted R_1 and R_2, and leakage fluxes, denoted ϕ_{l1} and ϕ_{l2} as shown in Figure 4.2, which link their own windings through air and can be modelled by leakage reactances X_1 and X_2.

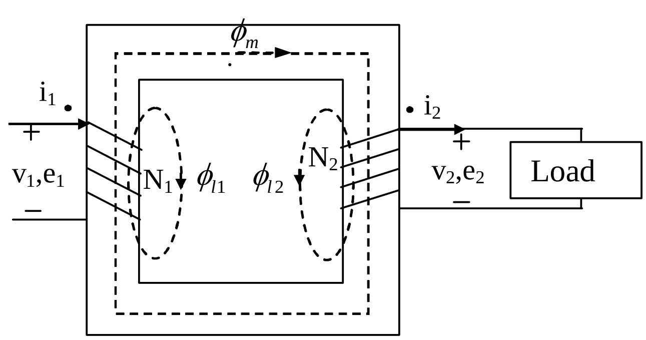


Figure 4.3 A practical transformer

The core of a practical transformer has finite permeability and core loss, so the primary winding draws the excitation current from the source even though there is no load attached to the secondary winding. The excitation current $\hat{I}_\phi$ is the sum of the core-loss current $\hat{I}_c$ and the magnetization current $\hat{I}_m$, that is,

$$\hat{I}_\phi = \hat{I}_c + \hat{I}_m$$

The core loss can be modelled by an equivalent core-loss resistance R_c and the magnetization effect can be described by an equivalent magnetizing reactance X_m. If the induced voltage across the primary winding is $\hat{E}_1$, then

$$\hat{I}_c = \frac{\hat{E}_1}{R_c}$$

$$\hat{I}_m = \frac{\hat{E}_1}{jX_m}$$

Note that the effective mutual flux created by $\hat{I}_\phi$ should be equal to the mutual flux in the core. Assume that the reluctance of the core is $\Re$. Then, we have

$$\phi = \frac{N_1\hat{I}_\phi}{\Re} = \frac{N_1\hat{I}_1 - N_2\hat{I}_2}{\Re}$$

that is,

$$N_1\hat{I}_\phi = N_1\hat{I}_1 - N_2\hat{I}_2$$

Therefore, one gets

$$\hat{I}'_2 = \hat{I}_1 - \hat{I}_\phi = \frac{N_2}{N_1}\hat{I}_2$$

where $\hat{I}'_2$ is the load current viewed from the primary side. Similar to the case of the ideal transformer, the relationship between the induced voltage in the primary and secondary sides is given by

$$\frac{\hat{E}_1}{\hat{E}_2} = \frac{N_1}{N_2} = a$$

which implies that the relationship among the quantities $\hat{E}_1, \hat{E}_2, \hat{I}'_2,$ and $\hat{I}_2$ can be modelled by an ideal transformer. The equivalent circuit for a practical transformer is shown in Figure 4.4.

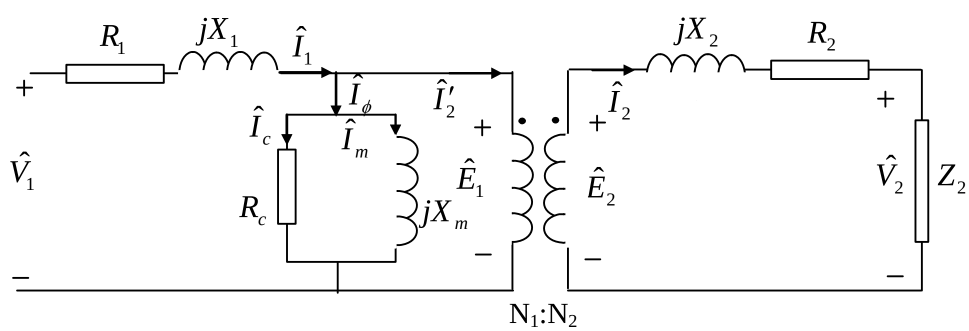


Figure 4.4 The equivalent circuit of a practical transformer

After the secondary is transformed to the primary side, the equivalent circuit becomes one as shown in Figure 4.5.

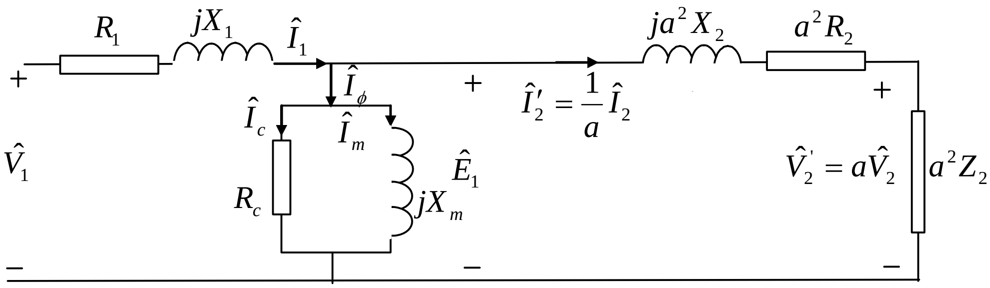


Figure 4.5 The equivalent circuit as viewed from the primary side

On the other hand, Figure 4.6 shows the equivalent circuit as viewed from the secondary side.

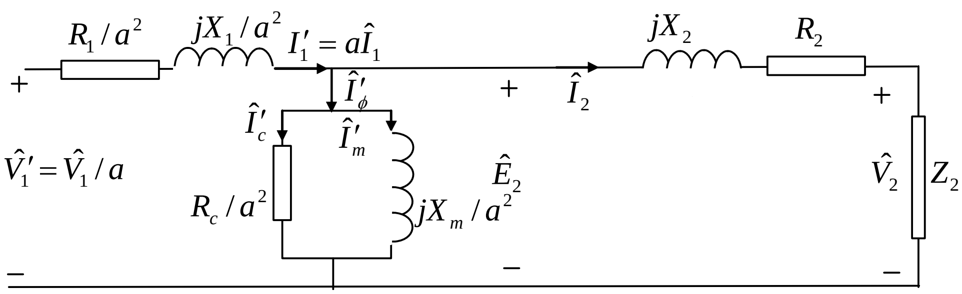


Figure 4.6 The equivalent circuit as viewed from the secondary side

In a well-designed transformer, $R_1, R_2, X_1,$ and X_2 are kept as small as possible, and R_c and X_m are kept as big as possible so that the transformer efficiency can be made as high as possible. Since R_1 and X_1 are quite low, the voltage drop across them is also low in comparison with the applied voltage. Without introducing any appreciable error, we can assume that the voltage across the parallel branch is the same as the applied voltage. This assumption allows us to move the parallel branch as shown in Figure 4.7.

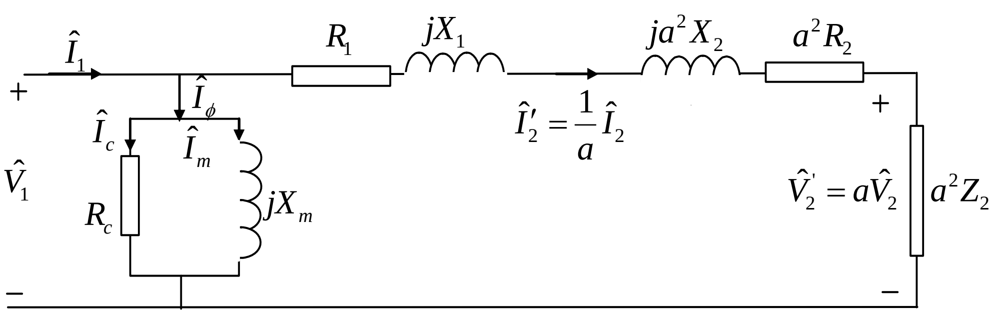


Figure 4.7 The approximate equivalent circuit as viewed from the

4.3 Voltage Regulation and Maximum Efficiency Criterion

The voltage regulation $VR\%$ is defined as

$$VR\% = \frac{V_{2NL} - V_{2FL}}{V_{2FL}} \times 100$$

where V_{2NL} and V_{2FL} are effective values of no-load and full-load voltages at the secondary terminals. For an ideal transformer, the voltage regulation is zero. The

smaller the voltage regulation, the better the transformer.
The input power to an transformer is calculated by

$$S_{in} = \left|\hat{V}_1 \hat{I}_1^*\right|$$

$$P_{in} = V_1 I_1 \cos\theta_1 = \mathrm{Re}\left[\hat{V}_1 \hat{I}_1^*\right]$$

$$Q_{in} = V_1 I_1 \sin\theta_1 = \mathrm{Im}\left[\hat{V}_1 \hat{I}_1^*\right]$$

where $cos\theta_1$ is the power factor of the transformer and θ_1 is the power angle of the transformer (Note that the power angle is the difference between the voltage phase angle and current phase angle).
The output power from an transformer is calculated by

$$S_{in} = \left|\hat{V}_2 \hat{I}_2^*\right|$$

$$P_{in} = V_2 I_2 \cos\theta 2 = \mathrm{Re}\left[\hat{V}_2 \hat{I}_2^*\right]$$

$$Q_{in} = V_2 I_2 \sin\theta_2 = \mathrm{Im}\left[\hat{V}_2 \hat{I}_2^*\right]$$

where $cos\theta_2$ is the power factor of the load and θ_2 is the power angle of the load.
It follows from the approximate equivalent circuit shown in Figure 4.7 that the output power can also be calculated by

$$P_o = I_2' V_2' \cos\theta_2$$

The copper loss is

$$P_{cu} = (I_2')^2 (R_1 + a^2 R_2)$$

Recall that the core loss is determined by $P_m = K_e f^2 B^2 + K_h f B^n$. The flux in the transformer is almost constant, so is B. Therefore, P_m is essentially constant. The input power can also determined by

$$P_{in} = P_o + P_{cu} + P_m = I_2' V_2' \cos\theta_2 + (I_2')^2 (R_1 + a^2 R_2) + P_m$$

The efficiency of the transformer is

$$\eta = \frac{P_o}{P_{in}} = \frac{I_2' V_2' \cos\theta_2}{I_2' V_2' \cos\theta_2 + (I_2')^2 (R_1 + a^2 R_2) + P_m}$$

which is a function of I_2'. To get the load current $I_{2\eta}'$ for the maximum efficiency, we differentiate η with respect with I_2' and set it to be zero, that is,

$$\frac{d\eta}{dI_p} = \frac{V_2' \cos\theta_2 \left(I_2' V_2' \cos\theta_2 + (I_2')^2 (R_1 + a^2 R_2) + P_m\right) - I_2' V_2' \cos\theta_2 (V_2' \cos\theta_2 + 2 I_2' (R_1 + a^2 R_2))}{\left(I_2' V_2' \cos\theta_2 + (I_2')^2 (R_1 + a^2 R_2) + P_m\right)^2}$$

$$= \frac{V_2' \cos\theta_2 \left(P_m - (I_2')^2 (R_1 + a^2 R_2)\right)}{\left(I_2' V_2' \cos\theta_2 + (I_2')^2 (R_1 + a^2 R_2) + P_m\right)^2} = 0$$

which implies that

$$P_m = (I_2')^2 (R_1 + a^2 R_2) = P_{cu}$$

The above equation indicates that the efficiency of a transformer is maximum when the copper loss is equal to the core loss. The load current $I_{2\eta}'$ for the maximum efficiency is given by

$$I_{2\eta}' = \sqrt{\frac{P_m}{R_1 + a^2 R_2}}$$

4.4 Determination of Transformer Parameters

Suppose a step-down transformer is tested in this section.

The Short-Circuit Test

Short-circuit the low-voltage side, increase the voltage on the high-voltage side until the rated current is reached on the low-voltage side, and measure the voltage, current, and power on the high-voltage side. The equivalent circuit for the short-circuit test is shown in Figure 4.8.

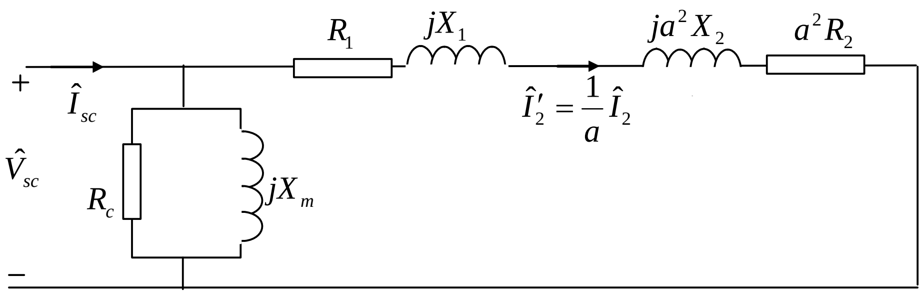


Figure 4.8 The equivalent circuit for the short-circuit test

Since the input voltage V_{sc} is very low, the excitation curcuit is small and can be neglected and so the parallel braches can be removed to simplify calculations. Now define $Z_{eq} = R_{eq} + jX_{eq} = R_1 + a^2R_2 + j(X_1 + a^2X_2)$. Then, it follows from the equivalent circuit that

$$R_{eq} = R_1 + a^2R_2 = \frac{P_{sc}}{I_{sc}^2}$$

$$|Z_{eq}| = \frac{V_{sc}}{I_{sc}}$$

$$X_{eq} = X_1 + a^2X_2 = \sqrt{|Z_{eq}|^2 - R_{eq}^2}$$

For most transformers, resistances and reactances can be separated by

$$R_1 = a^2R_2 = 0.5R_{eq}$$

$$X_1 = a^2X_2 = 0.5X_{eq}$$

The Open-Circuit Test

Open-circuit the high-voltage side, apply the rated voltage to the low-voltage side, and measure the voltage, current, and power on the low-voltage side. The equivalent circuit for the open-circuit test is shown in Figure 4.9.

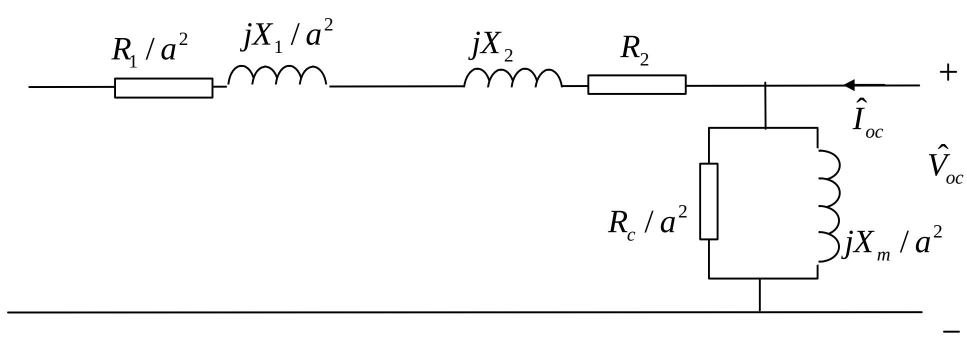


Figure 4.9 The equivalent circuit for the open-circuit test

Let $Z_\phi = \frac{1}{\frac{1}{R_c/a^2}+\frac{1}{jX_m/a^2}}$. Then it follows from the equivalent circuit that

$$R_c/a^2 = \frac{V_{oc}^2}{P_{oc}} \Rightarrow R_c = a^2\frac{V_{oc}^2}{P_{oc}}$$

$$|Z_\phi| = \frac{V_{oc}}{I_{oc}}$$

Note that $\frac{1}{Z_\phi} = \frac{1}{R_c/a^2} + \frac{1}{jX_m/a^2}$, i.e. $\left|\frac{1}{Z_\phi}\right|^2 = \left(\frac{1}{R_c/a^2}\right)^2 + \left(\frac{1}{X_m/a^2}\right)^2$. As a result, we have

$$X_m/a^2 = \frac{1}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2 - \left(\frac{1}{R_c/a^2}\right)^2}} \Rightarrow X_m = \frac{a^2}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2 - \left(\frac{1}{R_c/a^2}\right)^2}}$$

Example 4.1 A 50kVA 2400:240V transformer is tested and the following data were recorded: the short-circuit test readings with the low-voltage side short-circuited are 48V, 20.8A, and 617W; the open-circuit test readings with the high-voltage side open-circuited are 240V, 5.41A, and 186W. Find the transformer parameters, the efficiency, and the voltage regulation at full load and 0.8 power factor lagging. Determine the load current for the maximum efficiency.

Solution: The transformation ratio is $a = \frac{V_1}{V_2} = \frac{2400}{240} = 10$. The approximate equivalent circuit is shown in Figure 4.7. From the short-circuit test (see Figure 4.8),

$V_{sc} = 48V, I_{sc} = 20.8A, P_{sc} = 617W$

$$R_{eq} = \frac{P_{sc}}{I_{sc}^2} = \frac{617}{20.8^2} = 1.4261\Omega$$

$$|Z_{eq}| = \frac{V_{sc}}{I_{sc}} = \frac{48}{20.8} = 2.3077\Omega$$

$$X_{eq} = \sqrt{|Z_{eq}|^2 - R_{eq}^2} = \sqrt{2.3077^2 - 1.4261^2} = 1.8143\Omega$$

Therefore,

$$R_1 = 0.5R_{eq} = 0.5 \times 1.4261 = 0.71305\Omega$$

$$a^2R_2 = 0.5R_{eq} = 0.5 \times 1.4261 = 0.71305\Omega$$

$$X_1 = 0.5X_{eq} = 0.5 \times 1.8143 = 0.90715\Omega$$

$$a^2X_2 = 0.5X_{eq} = 0.5 \times 1.8143 = 0.90715\Omega$$

For the open-circuit test, the equivalent circuit is shown in Figure 4.9. From the open-circuit test, $V_{oc} = 240V,\ I_{oc} = 5.41A,\ P_{oc} = 186W$. Therefore,

$$R_c = a^2\frac{V_{oc}^2}{P_{oc}} = 10^2 \times \frac{240^2}{186} = 30968\Omega$$

$$|Z_\phi| = \frac{V_{oc}}{I_{oc}} = \frac{240}{5.41} = 44.362\Omega$$

$$X_m = \frac{a^2}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2 - \left(\frac{1}{R_c/a^2}\right)^2}} = \frac{10^2}{\sqrt{\left|\frac{1}{44.362}\right|^2 - \left(\frac{1}{30968/10^2}\right)^2}} = 4482.4\Omega$$

The full load current is

$$I_2 = \frac{S_2}{V_2} = \frac{50000}{240} = 208A$$

The load current referred to the primary side is

$$I_2' = \frac{1}{a}I_2 = \frac{1}{10}208.33 = 20.8A$$

The core loss is the same as the input power in the open-circuit test, that is,

$$P_m = 168W$$

The copper loss is

$$P_{cu} = (I_2')^2(R_1 + a^2R_2) = 20.8^2 \times 1.4261 = 617W$$

The output power at full load is

$$P_o = V_2 I_2 \cos\theta_2 = 240 \times 208 \times 0.8 = 39936W$$

The input power at full load is

$$P_{in} = P_o + P_{cu} + P_m = 39936 + 617 + 168 = 40721W$$

The efficiency at full load is

$$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{39936}{40721} \times 100 = 98.0\%$$

The load voltage phasor is chosen as a reference, that is, $\widehat{V}_2 = 240\angle 0^\circ V$ and $\widehat{V}_2' = a\widehat{V}_2 = 2400\angle 0^\circ V$. Then, the load current phasor is

$$\widehat{I}_2 = I_2 \angle \cos^{-1}(0.8) = 208\angle -36.9^\circ A$$

The load current phasor referred to the primary side is

$$\widehat{I}_2' = \frac{1}{a}\widehat{I}_2 = \frac{1}{10}208\angle -36.9^\circ = 20.8\angle -36.9^\circ = 20.8(0.8 - j0.6)A$$

It follows from Figure 4.6 that

$$\widehat{V}_1 = \widehat{V}_2' + \widehat{I}_2'(R_1 + a^2R_2 + j(X_1 + a^2X_2))$$
$$= 2400 + 20.8(0.8 - j0.6)(1.4261 + j1.8143) = 2446.3 + j12.392$$
$$= \sqrt{12.392^2 + 2446.3^2} \angle \frac{180}{\pi}\tan^{-1}\left(\frac{12.392}{2446.3}\right) = 2446.3\angle\ 0.29^\circ V$$

Now let us find the no load output voltage corresponding to $\widehat{V}_1 = 2446.3\angle\ 0.29^\circ V$ by using the approximate equivalent circuit. It is obvious that
$V_2' = V_1 = 2446.3$ and $V_2 = V_2'/a = 244.63V$. Therefore, the voltage regulation is

$$VR\% = \frac{244.63 - 240}{240} \times 100 = 1.93\%$$

The load current for the maximum efficiency viewed from the primary side is

$$I_{2\eta}' = \sqrt{\frac{P_m}{R_1 + a^2R_2}} = \sqrt{\frac{168}{1.4261}} = 10.854A$$

and the load current for the maximum efficiency is

$$I_{2\eta} = aI_{2\eta}' = 108.54A$$

4.5 Per-Unit Computations

quantities such as voltage, current, power, reactive power, volt-amperes, resistance, reactance, and impedance can be translated to and from per-unit form as follows:

$$\text{Quantity in per-unit} = \frac{\text{Actual quantity}}{\text{Base Value of quantity}}$$

For a single phase system, the base values must obey the electric circuit laws, that is,

$$P_{base}, Q_{base}, VA_{base} = V_{base}I_{base}$$
$$R_{base}, X_{base}, Z_{base} = \frac{V_{base}}{I_{base}}$$

1. Select a VA base and a base voltage at some point in the system.
2. Convert all quantities to per-unit.
3. Perform a standard electrical analysis with all quantities in per-unit.
4. Convert all quantities back to real units by multiplying their per-unit values by their corresponding base values.

Note that the turns ratio of an ideal transformer in per unit is one if V_1 and V_2 are chosen as the base voltages for the primary and secondary sides since

$$a_{pu} = \frac{V_{1,pu}}{V_{2,pu}} = \frac{\frac{V_1}{V_{1base}}}{\frac{V_2}{V_{2base}}} = \frac{\frac{V_1}{V_1}}{\frac{V_2}{V_2}} = 1$$

Example 4.2: A single-phase generator with an internal impedance $Z_g = 23 + j92m\Omega$ is connected to a load via a 46kVA, 230/2300V, step-up transformer, a short transmission line and a 46kVA, 2300/115V, step-down transformer. The impedance of the transmission line is $Z_{tl} = 2.07 + j4.14\Omega$. The parameters of step-up and step-down transformers are:

$Z_{1g} = 23 + j69m\Omega$, $Z_{\phi g} = 138 + j69\Omega$, $Z_{2g} = 2.3 + j6.9\Omega$, $Z_{1l} = 2.33 + j6.9\Omega$, $Z_{\phi l} = 11.5 + j9.2k\Omega$, $Z_{2l} = 5.75 + j17.25m\Omega$.

Determine (a) the generator voltage, (b) the generator current, and (c) the overall efficiency of the system at full load and 0.866 pf lagging.

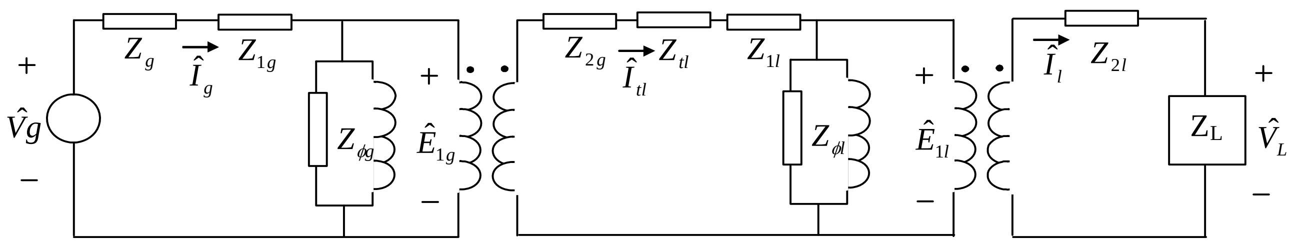


Figure 4.10 The circuit for Example 4.2

Solution:The equivalent circuit of the system incorporating ideal transformers is given in Figure 4.10.

For the generator side, choose the base values $V_{bg} = 230V$ and $S_{bg} = 46000VA$. Then, we have

$$I_{bg} = \frac{S_{bg}}{V_{bg}} = \frac{46000}{230} = 200A$$

$$Z_{bg} = \frac{V_{bg}}{I_{bg}} = \frac{230}{200} = 1.15\Omega$$

The per-unit impedance of the generator is

$$Z_{g,pu} = \frac{Z_g}{Z_{bg}} = \frac{0.023+j0.092}{1.15} = 0.02 + j0.08$$

The per-unit parameters on the primary side of the step-up transformer are

$$Z_{1g,pu} = \frac{Z_{1g}}{Z_{bg}} = \frac{0.023+j0.069}{1.15} = 0.02 + j0.06$$

$$Z_{\phi g,pu} = \frac{Z_{\phi g}}{Z_{bg}} = \frac{138+j69}{1.15} = 120 + j60$$

For the transmission line side, choose the base values $V_{btl} = 2300V$ and $S_{btl} = 46000VA$. Then, we have

$$I_{btl} = \frac{S_{btl}}{V_{btl}} = \frac{46000}{2300} = 20A$$

$$Z_{btl} = \frac{V_{btl}}{I_{btl}} = \frac{2300}{20} = 115\Omega$$

The per-unit impedance on the secondary side of the step-up transformer is

$$Z_{2g,pu} = \frac{Z_{2g}}{Z_{btl}} = \frac{2.3+j6.9}{115} = 0.02 + j0.06$$

The per-unit impedance of the transmission line is

$$Z_{tl,pu} = \frac{Z_{tl}}{Z_{btl}} = \frac{2.07+j4.14}{115} = 0.018 + j0.036$$

The per-unit parameters on the primary side of the step-down transformer are

$$Z_{1l,pu} = \frac{Z_{1l}}{Z_{btl}} = \frac{2.3+j6.9}{115} = 0.02 + j0.06$$
$$Z_{\phi l,pu} = \frac{Z_{\phi l}}{Z_{btl}} = \frac{11500+j9200}{115} = 100 + j80$$

For the load side, choose the base values $V_{bl} = 115V$ and $S_{bl} = 46000VA$. Then, we have

$$I_{bl} = \frac{S_{bl}}{V_{bl}} = \frac{46000}{115} = 400A$$
$$Z_{bl} = \frac{V_{bl}}{I_{bl}} = \frac{115}{400} = 0.2875\Omega$$

The per-unit impedance on the secondary side of the step-down transformer is

$$Z_{2l,pu} = \frac{Z_{2l}}{Z_{bl}} = \frac{0.00575+j0.01725}{0.2875} = 0.02 + j0.06$$

The per-unit load voltage and per-unit load current are

$$V_{l,pu} = \frac{V_l}{V_{bl}} = \frac{115}{115} = 1$$
$$I_{l,pu} = \frac{I_l}{I_{bl}} = \frac{\frac{S_l}{V_l}}{400} = \frac{\frac{46000}{115}}{400} = 1$$

The load voltage and current phasors are

$$\widehat{V}_{l,pu} = 1\angle 0^\circ$$
$$\widehat{I}_{l,pu} = 1\angle -30^\circ$$

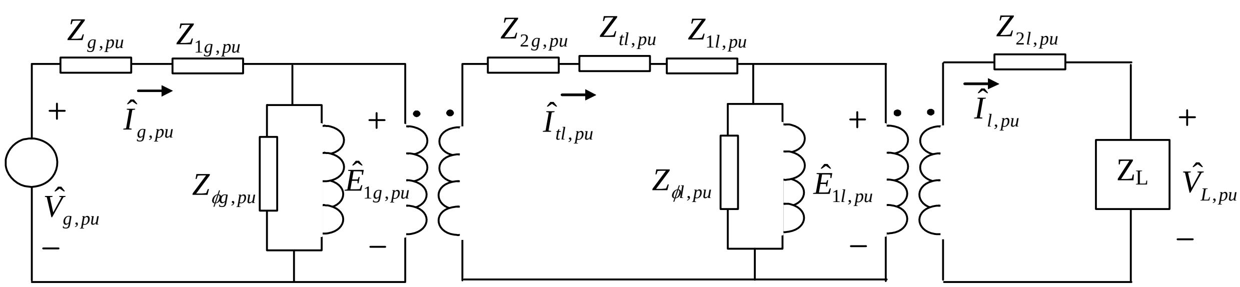


Figure 4.11 The equivalent circuit in per-unit for Example 4.2

The equivalent circuit of the system in per unit is shown in Figure 4.11. It follows from this equivalent circuit that

$$\widehat{E}_{1l,pu} = \widehat{I}_{l,pu}Z_{2l,pu} + \widehat{V}_{l,pu} = (1\angle -30^\circ)(0.02 + j0.06) + 1\angle 0^\circ$$
$$= \left(\cos\left(\tfrac{\pi}{6}\right) - j\sin\left(\tfrac{\pi}{6}\right)\right)(0.02 + j0.06) + 1 = 1.047 + j0.042$$
$$\widehat{I}_{tl,pu} = \widehat{I}_{l,pu} + \frac{\widehat{E}_{1l,pu}}{Z_{\phi l,pu}} = 1\angle -30^\circ + \frac{1.047+j0.042}{100+j80}$$
$$= \cos\left(\tfrac{\pi}{6}\right) - j\sin\left(\tfrac{\pi}{6}\right) + \frac{(1.047+j0.042)(100-j80)}{100^2+80^2} = 0.872 - j0.505$$
$$\widehat{E}_{1g,pu} = \widehat{I}_{tl,pu}(Z_{2g,pu} + Z_{tl,pu} + Z_{1l,pu}) + \widehat{E}_{1l,pu}$$
$$= (0.872 - j0.505)(0.02 + j0.06 + 0.018 + j0.036 + 0.02 + j0.06) + 1.047 + j0.042$$
$$= 1.176 + j0.149$$
$$\widehat{I}_{g,pu} = \widehat{I}_{tl,pu} + \frac{\widehat{E}_{1g,pu}}{Z_{\phi g,pu}} = 0.872 - j0.505 + \frac{1.176+j0.149}{120+j60}$$
$$= 0.872 - j0.505 + \frac{(1.176+j0.149)(120-j60)}{120^2+60^2}$$
$$= 0.880 - j0.508$$
$$\widehat{V}_{g,pu} = \widehat{I}_{g,pu}(Z_{g,pu} + Z_{1g,pu}) + \widehat{E}_{1g,pu}$$

$$= (0.880 - j0.508)(0.02 + j0.08 + 0.02 + j0.06) + 1.176 + j0.149$$
$$= 1.282 + j0.252$$

(a) Therefore, the generator voltage is

$$\hat{V}_g = V_{bg}\hat{V}_{g,pu} = 230(1.282 + j0.252)$$
$$= 294.86 + j57.96 = \sqrt{294.86^2 + 57.96^2} \angle \tan^{-1}\left(\tfrac{57.96}{294.86}\right)\tfrac{180}{\pi}$$
$$= 300.5\angle 11.1^\circ V$$

(b) The generator current is

$$\hat{I}_g = I_{bg}\hat{I}_{g,pu} = 200(0.880 - j0.508)$$
$$= 176.0 - j101.6 = \sqrt{176.0^2 + 101.6^2} \angle \tan^{-1}\left(\tfrac{-101.6}{176.0}\right)\tfrac{180}{\pi}$$
$$= 203.2\angle -30^\circ A$$

(c) On a per-unit basis, the rated power output at a 0.866 pf lagging is

$$P_{o,pu} = \hat{V}_{l,pu}\hat{I}_{l,pu}\cos\theta = 0.866$$

The per-unit input power from the generator is

$$P_{in,pu} = \mathrm{Re}\left[\hat{V}_{g,pu}\hat{I}^*_{g,pu}\right] = \mathrm{Re}[(1.282 + j0.252)(0.880 + j0.508)] = 1.0002$$

Thus, the efficiency is

$$\eta = \tfrac{P_{o,pu}}{P_{in,pu}} \times 100 = \tfrac{0.866}{1.0002} \times 100 = 86.6\%$$

4.6 Autotransformers

An ideal two-winding transformer can be connected as an ideal autotransformer. There are four possible ways to connect a two-winding transformer as an autotransformer, as shown in Figures 4.12-4.15.

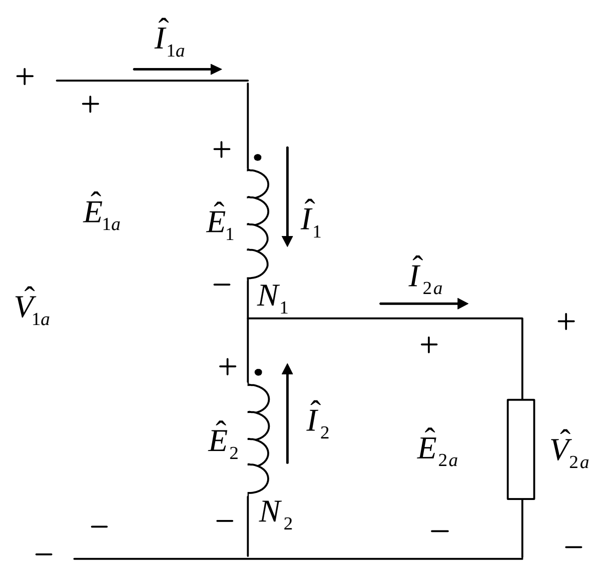


Figure 4.12 A step-down authotransformer

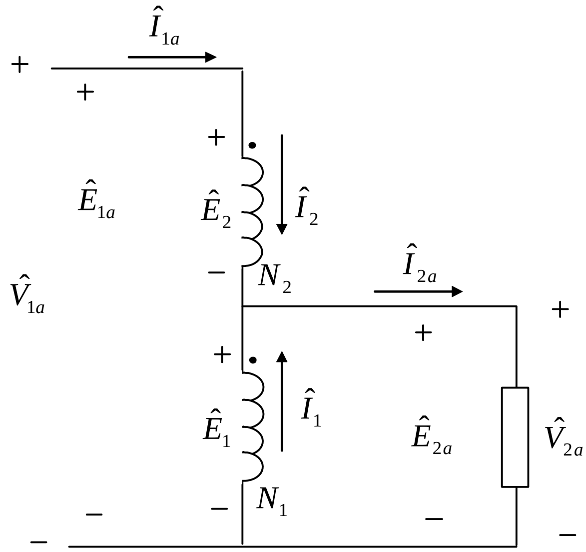


Figure 4.13 A step-down authotransformer

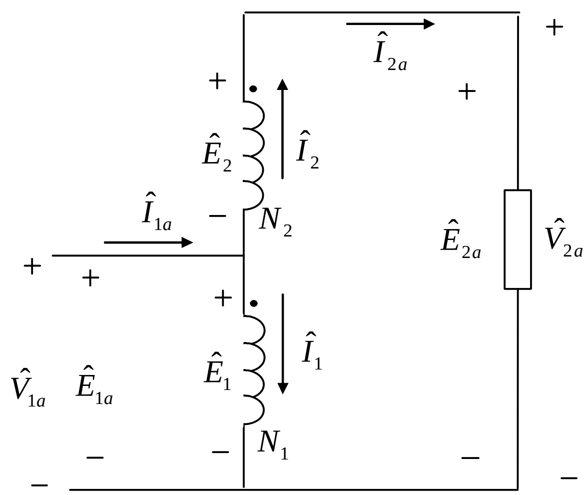


Figure 4.14 A step-up autotransformer

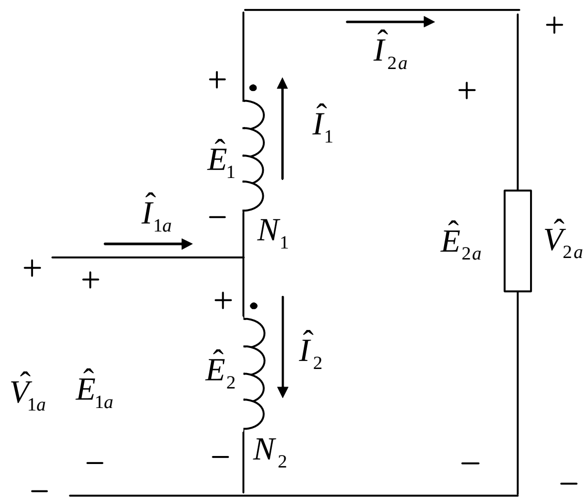


Figure 4.15 A step-up autotransformer

The following examples show how to calculate the primary winding voltage and current, $\hat{V}_{1a}$ and $\hat{I}_{1a}$, the secondary winding voltage and current, $\hat{V}_{2a}$ and $\hat{I}_{2a}$, the ratio of transformation a_T, and the apparent power input and output, $\hat{S}_{ina}$ and $\hat{S}_{oa}$.

Example 4.3: A 24kVA 2400/240V two-winding transformer is to be connected as an autotransformer. For each possible combination, determine the primary winding voltage and current, V_{1a} and I_{1a}, the secondary winding voltage and current, V_{2a} and I_{2a}, the ratio of transformation a_T, and the apparent power input and output, S_{ina} and S_{oa} under ideal conditions.

Solution: For the given information for the two-winding transformer, we get

$$E_1 = V_1 = 2400V, E_2 = V_2 = 240V,$$
$$a = \frac{V_1}{V_2} = 10, S_o = 24000VA,$$
$$I_2 = \frac{S_o}{V_2} = 100A, I_1 = \frac{I_2}{a} = 10A$$

For the autotransformer shown in Figure 4.12,

$$V_{1a} = E_{1a} = E_1 + E_2 = 2640V, V_{2a} = E_{2a} = E_2 = 240V,$$
$$a_T = \frac{V_{1a}}{V_{2a}} = 11, I_{1a} = I_1 = 10A, I_{2a} = I_1 + I_2 = 110A,$$
$$S_{ina} = V_{1a}I_{1a} = 2640 \times 10 = 26400VA, S_{oa} = V_{2a}I_{2a} = 240 \times 110 = 26400VA$$

The nominal rating of the autotransformer in Figure 4.12 is 26.4kVA, 2640/240V.

For the autotransformer shown in Figure 4.13,

$$V_{1a} = E_{1a} = E_1 + E_2 = 2640V, V_{2a} = E_{2a} = E_1 = 2400V,$$
$$a_T = \frac{V_{1a}}{V_{2a}} = \frac{2640}{2400} = 1.1, I_{1a} = I_2 = 100A, I_{2a} = I_1 + I_2 = 110A,$$
$$S_{ina} = V_{1a}I_{1a} = 2400 \times 100 = 264000VA, S_{oa} = V_{2a}I_{2a} = 2400 \times 110 = 264000VA$$

The nominal rating of the autotransformer in Figure 4.13 is 264kVA, 2640/2400V.

For the autotransformer shown in Figure 4.14,

$$V_{1a} = E_{1a} = E_1 = 2400V, V_{2a} = E_{2a} = E_1 + E_2 = 2640V,$$
$$a_T = \frac{V_{1a}}{V_{2a}} = \frac{2400}{2640} = 0.91, I_{1a} = I_1 + I_2 = 110A, I_{2a} = I_2 = 100A,$$
$$S_{ina} = V_{1a}I_{1a} = 2400 \times 110 = 264000VA, S_{oa} = V_{2a}I_{2a} = 2640 \times 100 = 264000VA$$

The nominal rating of the autotransformer in Figure 4.14 is 264kVA, 2400/2640V.

For the autotransformer shown in Figure 4.15,

$V_{1a} = E_{1a} = E_2 = 240V, V_{2a} = E_{2a} = E_1 + E_2 = 2640V,$

$a_T = \frac{V_{1a}}{V_{2a}} = \frac{240}{2640} = 0.091, I_{1a} = I_1 + I_2 = 110A, I_{2a} = I_1 = 10A,$

$S_{ina} = V_{1a}I_{1a} = 240 \times 110 = 26400VA, S_{oa} = V_{2a}I_{2a} = 2640 \times 10 = 26400VA$

The nominal rating of the autotransformer in Figure 4.15 is 26.4kVA, 240/2640V.

Note that the nominal rating of the autotransformer in Figure 4.13 or 4.14 is 10 times the nominal rating of the two-winding transformer.

Example 4.4: A 720VA 360/120V two-winding transformer has the following parameters: $R_1 = 18.9\Omega$, $X_1 = 21.6\Omega$, $R_2 = 2.1\Omega$, $X_2 = 2.4\Omega$, $R_{c1} = 8.64k\Omega$, $X_{m1} = 6.84k\Omega$. The transformer is connected as a 120/480V step-up autotransformer. If the autotransformer delivers the full load at 0.707 pf leading, determine its efficiency and voltage regulation.

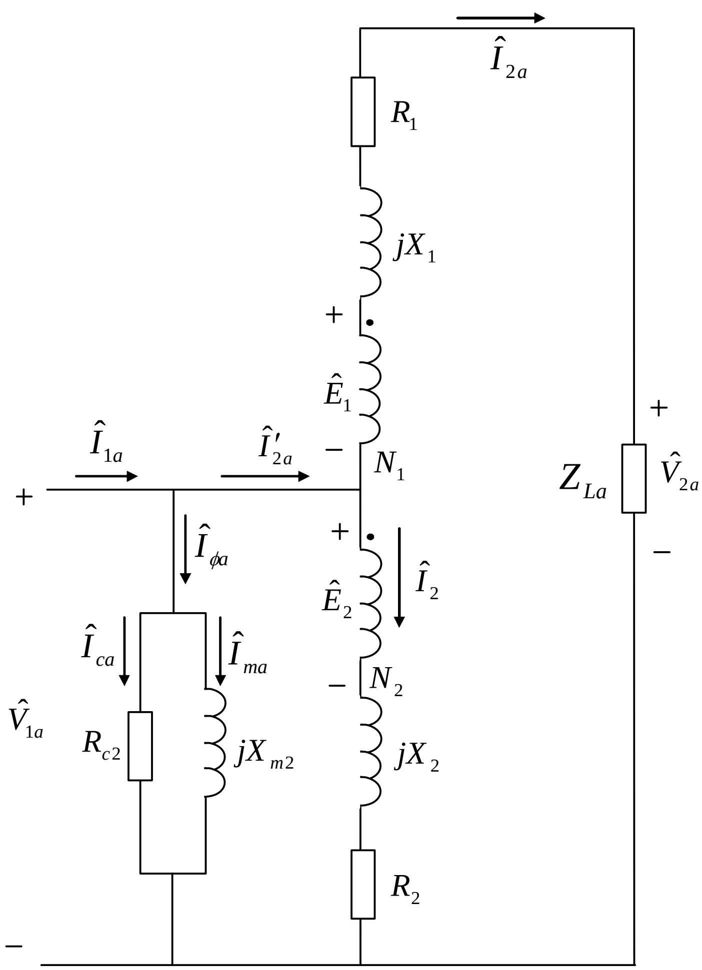


Figure 4.16 The approximate equivalent circuit of a

Solution: The turns ration of the two-winding transformer is $a = \frac{360}{120} = 3$ and the turns ratio of the autotransformer is $a_T = \frac{120}{480} = 0.25$. The equivalent core-loss resistance and the magnetizing reactance on the secondary side of the two-winding transformer is

$$R_{c2} = \frac{R_{c1}}{a^2} = \frac{8640}{3^2} = 960\Omega,\ X_{m2} = \frac{X_{m1}}{a^2} = \frac{6840}{3^2} = 760\Omega$$

The approximate equivalent circuit of the autotransformer is shown in Figure 4.16. Assume that $\widehat{V}_{2a} = 480\angle 0^\circ V$. The full load current is

$$I_{2a} = I_1 = \frac{S_{in}}{V_1} = \frac{720}{360} = 2A, \text{ so } \widehat{I}_{2a} = 2\angle 45^\circ A = 2\cos\left(45\frac{\pi}{180}\right) + j2\sin\left(45\frac{\pi}{180}\right)$$

and $\widehat{I}'_{2a} = \frac{\widehat{I}_{2a}}{a_T} = \frac{2}{0.25} = 8\angle 45^\circ A = 8\cos\left(45\frac{\pi}{180}\right) + j8\sin\left(45\frac{\pi}{180}\right)$.

Hence,

$$\begin{aligned}\widehat{I}_2 &= \widehat{I}'_{2a} - \widehat{I}_{2a} = 8\angle 45^\circ - 2\angle 45^\circ \\ &= 8\cos\left(45\frac{\pi}{180}\right) + j8\sin\left(45\frac{\pi}{180}\right) - 2\cos\left(45\frac{\pi}{180}\right) - j2\sin\left(45\frac{\pi}{180}\right) \\ &= 6\cos\left(45\frac{\pi}{180}\right) + j6\sin\left(45\frac{\pi}{180}\right)\end{aligned}$$

Note that $\widehat{E}_1 = a\widehat{E}_2 = 3\widehat{E}_2$. Then, it follows from KVL that

$$\widehat{E}_1 - \widehat{I}_{2a}(R_1 + jX_1) - \widehat{V}_{2a} + \widehat{I}_2(R_2 + jX_2) + \widehat{E}_2 = 0$$

or

$$\begin{aligned}\widehat{E}_2 &= \frac{1}{4}\left(\widehat{I}_{2a}(R_1 + jX_1) + \widehat{V}_{2a} - \widehat{I}_2(R_2 + jX_2)\right) \\ &= \frac{1}{4}\left(2\cos\left(45\frac{\pi}{180}\right) + j2\sin\left(45\frac{\pi}{180}\right)\right)(18.9 + j21.6) \\ &+ \frac{480}{4} - \frac{1}{4}\left(6\cos\left(45\frac{\pi}{180}\right) + j6\sin\left(45\frac{\pi}{180}\right)\right)(2.1 + j2.4) \\ &= 119.36 + j9.5459V\end{aligned}$$

Thus,

$$\begin{aligned}\widehat{V}_{1a} &= \widehat{E}_2 + \widehat{I}_2(R_2 + jX_2) \\ &= 119.36 + j9.5459 + \left(6\cos\left(45\frac{\pi}{180}\right) + j6\sin\left(45\frac{\pi}{180}\right)\right)(2.1 + j2.4) \\ &= 118.09 + j28.638 = \sqrt{118.09^2 + 28.638^2}\angle \tan^{-1}\left(\frac{28.638}{118.09}\right)\frac{180}{\pi} = 121.51\angle 13.6^\circ V\end{aligned}$$

On the other hand,

$$\begin{aligned}\widehat{I}_{1a} &= \widehat{I}_{\phi a} + \widehat{I}'_{2a} = \frac{\widehat{V}_{1a}}{R_{c2}} + \frac{\widehat{V}_{1a}}{jX_{m2}} + \widehat{I}'_{2a} = \widehat{V}_{1a}\left(\frac{1}{R_{c2}} + \frac{1}{jX_{m2}}\right) + \widehat{I}'_{2a} \\ &= (118.09 + j28.638)\left(\frac{1}{960} + \frac{1}{j760}\right) + 8\cos\left(45\frac{\pi}{180}\right) + j8\sin\left(45\frac{\pi}{180}\right) \\ &= (118.09 + j28.638)\left(\frac{1}{960} - j\frac{1}{760}\right) + 8\cos\left(45\frac{\pi}{180}\right) + j8\sin\left(45\frac{\pi}{180}\right) \\ &= 5.8176 + j5.5313\end{aligned}$$

Therefore,

$$P_o = \mathrm{Re}\left(\widehat{V}_{2a}\widehat{I}^*_{2a}\right) = \mathrm{Re}\left((480)\left(2\cos\left(45\frac{\pi}{180}\right) - j2\sin\left(45\frac{\pi}{180}\right)\right)\right) = 678.82W$$

$$P_{in} = \mathrm{Re}\left(\widehat{V}_{1a}\widehat{I}^*_{1a}\right) = \mathrm{Re}((118.09 + j28.638)(5.8176 - j5.5313)) = 845.41W$$

$$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{678.82}{845.41} \times 100 = 80.3\%$$

If we remove the load, the no-load voltage at the secondary of the autotransformer is

$$V_{2aNL} = \frac{V_{1a}}{a_T} = \frac{121.51}{0.25} = 486.04V$$

We now can compute the voltage regulation as

$$VR\% = \frac{V_{2aNL} - V_{2aFL}}{V_{2aFL}} \times 100 = \frac{486.04 - 480}{480} \times 100 = 1.24\%$$

4.7 Three-Phase Transformers

The three windings on either side of a three-phase transformer can be connected either in Y or in △. Therefore, a three-phase transformer can be connected in four possible ways: Y/Y, △/△, △/Y, Y/△, as shown in Figures 17-20.

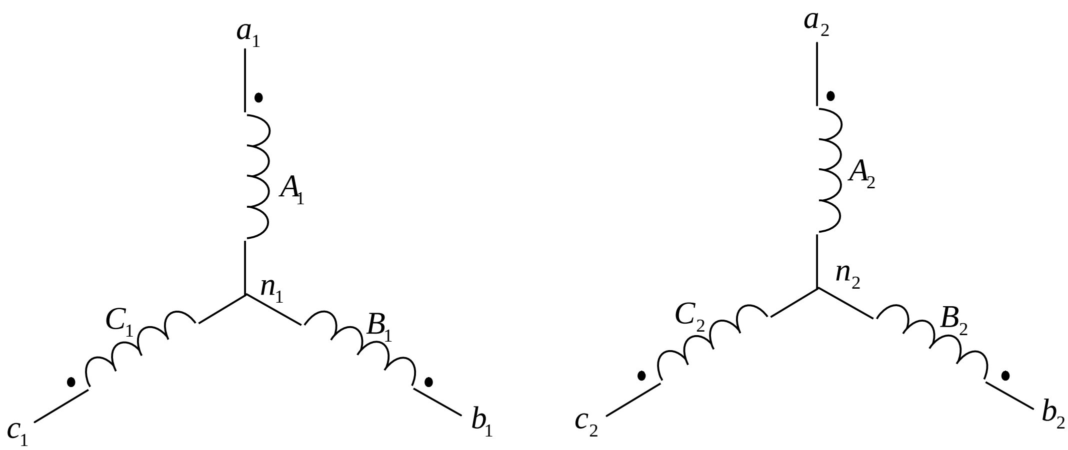


Figure 4.17 Y-Y connection

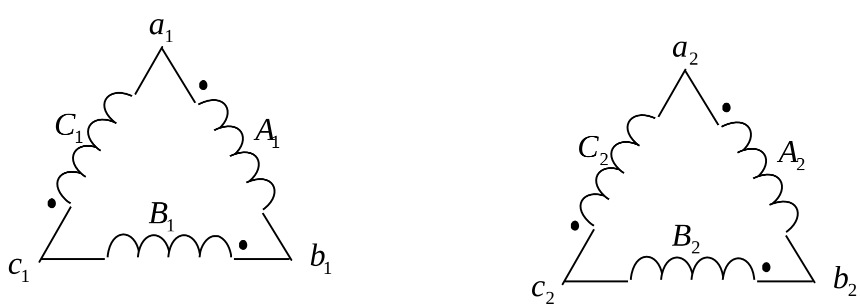


Figure 4.18 △ – △ connection

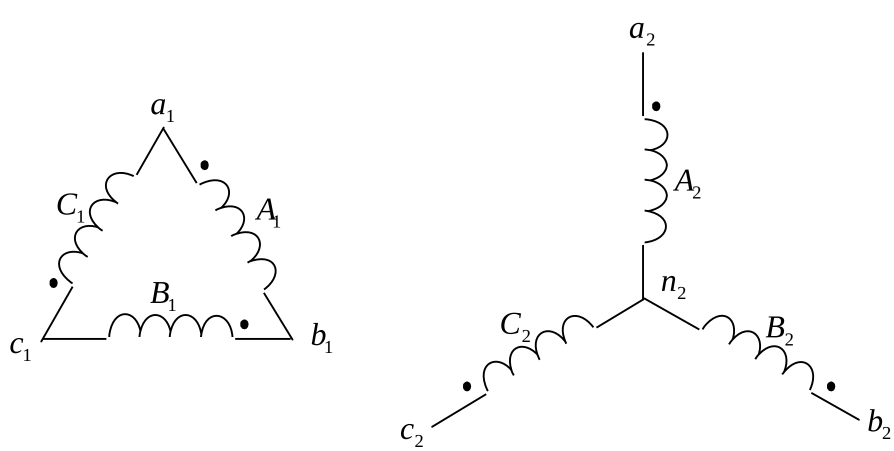


Figure 4.19 △ – Y connection

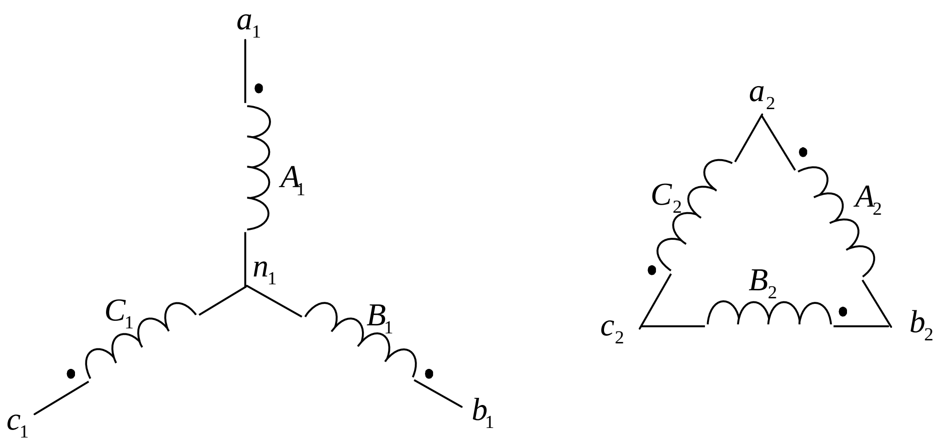


Figure 4.20 Y-△connection

For simplicity, we assume that the three-phase transformer delivers a balanced load. Under steady-state conditions, a three-phase transformer can be analyzed by (a) transforming a △-connected winding to a Y-connected winding using △-to-Y transformation, (b) drawing a per-phase equivalent circuit, and (c) computing quantities for the per-phase equivalent circuit.

△-to-Y transformation: If $Z_\triangle$ is the impedance in a △-connected winding, the equivalent impedance Z_Y in a Y-connected winding is

$$Z_Y = \frac{1}{3} Z_\triangle$$

and

$$\hat{V}_{an} = \frac{\hat{V}_{ab}}{\sqrt{3}} \angle -30^\circ$$

$$\hat{I}_{A'} = \sqrt{3}\,\hat{I}_A \angle -30^\circ$$

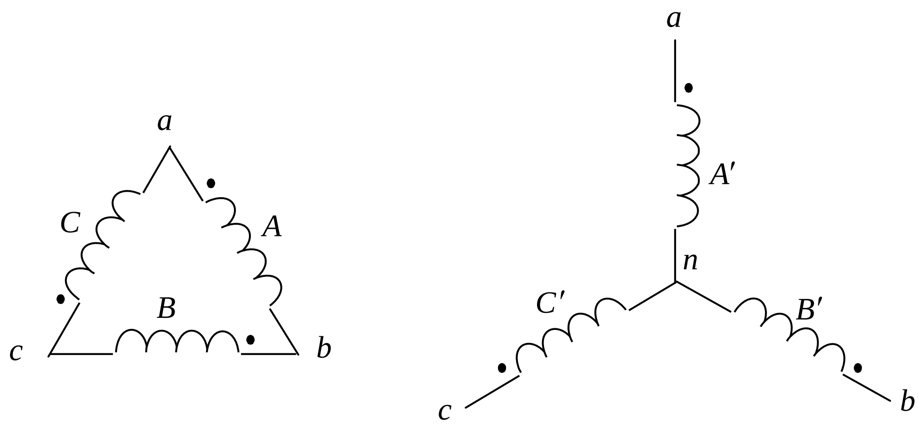
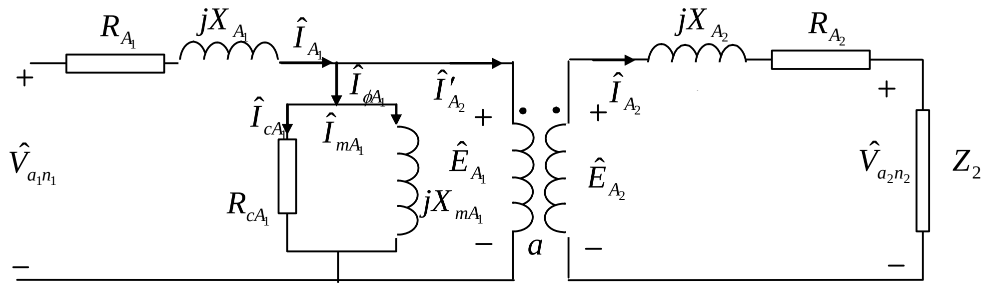


Figure 4.21 $\triangle - Y$ transformation

Example 4.5: A three-phase transformer is assembled by connecting three 720VA 360/120V single-phase transformers. The constants for each transformer are $R_H = 18.9\Omega$, $X_H = 21.6\Omega$, $R_L = 2.1\Omega$, $X_L = 2.4\Omega$, $R_{cH} = 8.64k\Omega$, $X_{mH} = 6.84k\Omega$. For each of the four configurations, determine the nominal voltage and power ratings of the three-phase transformer. Draw the per-phase equivalent circuit for each configuration.



Solution: The power rating of the three-phase transformer for each configuration is $S_{3\phi} = 3 \times 720 = 2160VA$.

(**a**) **Y-Y connection**: The nominal values of the line voltages on the primary and the secondary sides are

$$V_{1L} = \sqrt{3}\,V_{a_1n_1} = \sqrt{3} \times 360 = 623.54V$$
$$V_{2L} = \sqrt{3}\,V_{a_2n_2} = \sqrt{3} \times 120 = 207.85V$$

Thus, the nominal ratings of the three-phase transformer are 2.16kVA 624/208V Y/Y connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$$a = \frac{360}{120} = 3,$$
$$R_{A_1} = R_H = 18.9\Omega,\ X_{A_1} = X_H = 21.6\Omega,$$
$$R_{A_2} = R_L = 2.1\Omega,\ X_{A_2} = X_L = 2.4\Omega,$$
$$R_{cA_1} = R_{cH} = 8.64k\Omega,\ X_{mA_1} = X_{mH} = 6.84k\Omega$$

(**b**) △-△ **connection**: The nominal values of the line voltages on the primary and the secondary sides are

$$V_{1L} = V_{a_1b_1} = 360V$$
$$V_{2L} = V_{a_2b_2} = 120V$$

Thus, the nominal ratings of the three-phase transformer are 2.16kVA 360/120V △/△ connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$$a = \frac{\frac{360}{\sqrt{3}}}{\frac{120}{\sqrt{3}}} = 3,$$
$$R_{A_1} = \frac{R_H}{3} = \frac{18.9}{3} = 6.3\Omega,\ X_{A_1} = \frac{X_H}{3} = \frac{21.6}{3} = 7.2\Omega,$$
$$R_{A_2} = \frac{R_L}{3} = \frac{2.1}{3} = 0.7\Omega,\ X_{A_2} = \frac{X_L}{3} = \frac{2.4}{3} = 0.8\Omega,$$
$$R_{cA_1} = \frac{R_{cH}}{3} = \frac{8.64}{3} = 2.88k\Omega,\ X_{mA_1} = \frac{X_{mH}}{3} = \frac{6.84}{3} = 2.28k\Omega$$

(**c**) △-**Y connection**: The nominal values of the line voltages on the primary and the secondary sides are

$$V_{1L} = V_{a_1b_1} = 360V$$
$$V_{2L} = \sqrt{3}\,V_{a_2n_2} = \sqrt{3} \times 120 = 207.85V$$

Thus, the nominal ratings of the three-phase transformer are 2.16kVA 360/208V △/Y connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$$a = \frac{\frac{360}{\sqrt{3}}}{120} = 1.732,$$
$$R_{A_1} = \frac{R_H}{3} = \frac{18.9}{3} = 6.3\Omega,\ X_{A_1} = \frac{X_H}{3} = \frac{21.6}{3} = 7.2\Omega,$$
$$R_{A_2} = R_L = 2.1\Omega,\ X_{A_2} = X_L = 2.4\Omega,$$
$$R_{cA_1} = \frac{R_{cH}}{3} = \frac{8.64}{3} = 2.88k\Omega,\ X_{mA_1} = \frac{X_{mH}}{3} = \frac{6.84}{3} = 2.28k\Omega,$$
$$\widehat{E}_{A_1} = a\widehat{E}_{A_2}\angle -30^\circ,\ \widehat{I}'_{A_2} = \frac{1}{a}\widehat{I}_{A_2}\angle -30^\circ$$

(d) Y-△ connection: The nominal values of the line voltages on the primary and the secondary sides are

$$V_{1L} = \sqrt{3}\,V_{a_1n_1} = \sqrt{3} \times 360 = 623.54V$$
$$V_{2L} = V_{a_2b_2} = 120V$$

Thus, the nominal ratings of the three-phase transformer are 2.16kVA 624/120V **Y**/△ connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$$a = \frac{360}{\frac{120}{\sqrt{3}}} = 5.196,$$
$$R_{A_1} = R_H = 18.9\Omega,\ X_{A_1} = X_H = 21.6\Omega,$$
$$R_{A_2} = \frac{R_L}{3} = \frac{2.1}{3} = 0.7\Omega,\ X_{A_2} = \frac{X_L}{3} = \frac{2.4}{3} = 0.8\Omega,$$
$$R_{cA_1} = R_{cH} = 8.64k\Omega,\ X_{mA_1} = X_{mH} = 6.84k\Omega,$$
$$\widehat{E}_{A_1} = a\widehat{E}_{A_2}\angle 30^\circ,\ \widehat{I}'_{A_2} = \frac{1}{a}\widehat{I}_{A_2}\angle 30^\circ$$

Example 4.6: Three single-phase transformers, each rated at 12kVA 120/240V 60Hz are connected to form a three-phase step-up Y/△ connection. The parameters of each transformer are $R_H = 133.5m\Omega$, $X_H = 201m\Omega$, $R_L = 39.5m\Omega$, $X_L = 61.5m\Omega$, $R_{cL} = 240\Omega$, $X_{mL} = 290\Omega$. What are the nominal voltage, current, and power ratings of the three-phase transformer. When it delivers the rated load at rated voltage and 0.8 pf lagging, determine the line voltages, the line currents, and the efficiency of the three-phase transformer.

Solution: The nominal values of the three-phase transformer are

$$S_{3\phi} = 3S_{1\phi} = 36kVA$$
$$V_{1\phi} = V_{a_1n_1} = 120V$$
$$V_{1L} = \sqrt{3}\,V_{a_1n_1} = \sqrt{3} \times 120 = 208V$$
$$V_{2\phi} = V_{a_2b_2} = 240V$$
$$V_{2L} = V_{a_2b_2} = 240V$$

For the equivalent Y/Y connection, the nominal values of the three-phase transformer are

$$V_{1\phi} = V_{a_1n_1} = 120V$$
$$V_{1L} = \sqrt{3}\,V_{a_1n_1} = \sqrt{3} \times 120 = 208V$$
$$V_{2\phi} = \frac{V_{a_2b_2}}{\sqrt{3}} = \frac{240}{\sqrt{3}} = 138.56V$$
$$V_{2L} = V_{a_2b_2} = 240V$$

Thus, the nominal ratings of the three-phase transformer are 36kVA 208/240V **Y**/△ connection. The per-phase equivalent circuit is shown in Figure 4.22 with the following parameters:

$a = \frac{120}{138.56} = 0.866$,

$R_{A_1} = R_L = 39.5 m\Omega$, $X_{A_1} = X_L = 61.5 m\Omega$,

$R_{A_2} = \frac{R_H}{3} = \frac{133.5}{3} = 44.5 m\Omega$, $X_{A_2} = \frac{X_H}{3} = \frac{201}{3} = 67 m\Omega$,

$R_{cA_1} = R_{cL} = 240\Omega$, $X_{mA_1} = X_{mL} = 290\Omega$,

$\widehat{E}_{A_1} = a\widehat{E}_{A_2}\angle 30^\circ$, $\widehat{I}'_{A_2} = \frac{\widehat{I}_{A_2}}{a}\angle 30^\circ$

Assuming the rated load voltage on a per-phase basis for the equivalent Y/Y connection as the reference, then

$\widehat{V}_{a_2n_2} = 138.56\angle 0^\circ V$

For a 0.8 lagging power factor, the load current is

$\widehat{I}_{A_2} = \frac{S_{1\phi}}{V_{2\phi}}\angle -\cos^{-1}(0.8) = \frac{12000}{138.56}\angle -\cos^{-1}(0.8) = 86.6\angle -36.87^\circ A$

The per-phase load current in the primary winding is

$\widehat{I}'_{A_2} = \frac{\widehat{I}_{A_2}}{a}\angle 30^\circ = \frac{86.6}{0.866}\angle(30^\circ - 36.87^\circ) = 100\angle -6.87^\circ A$

The per-phase voltage induced in the equivalent Y-connected secondary winding is

$\widehat{E}_{A_2} = \widehat{V}_{a_2n_2} + \widehat{I}_{A_2}(R_{A_2} + jX_{A_2}) = 138.56 + 86.6\angle -36.87^\circ(0.0445 + j0.067)$

$= 138.56 + 86.6(0.8 - j0.6)(0.0445 + j0.067)$

$= 145.12 + j2.3295 = \sqrt{145.12^2 + 2.3295^2}\angle \tan^{-1}\left(\frac{2.3295}{145.12}\right)\frac{180}{\pi} = 145.147\angle 0.92^\circ V$

The induced emf in the Y-connected primary winding is

$\widehat{E}_{A_1} = a\widehat{E}_{A_2}\angle 30^\circ = (0.866\angle 30^\circ)(145.147\angle 0.92^\circ) = 0.866 \times 145.$
$147\angle(0.92^\circ + 30^\circ) = 125.7\angle 30.92^\circ V$

The excitation current is

$\widehat{I}_{\phi A_1} = \frac{\widehat{E}_{A_1}}{\frac{1}{\frac{1}{R_{cA_1}} + \frac{1}{jX_{mA_1}}}} = \widehat{E}_{A_1}\left(\frac{1}{R_{cA_1}} + \frac{1}{jX_{mA_1}}\right) = (125.7\angle 30.92^\circ)\left(\frac{1}{240} + \frac{1}{j290}\right)$

$= \left(125.7\cos\left(30.92\frac{\pi}{180}\right) + j125.7\sin\left(30.92\frac{\pi}{180}\right)\right)\left(\frac{1}{240} - j\frac{1}{290}\right) = 0.67204 - j0.10272 A$

Thus, the primary current is

$\widehat{I}_{A_1} = \widehat{I}_{\phi A_1} + \widehat{I}'_{A_2} = 0.67204 - j0.10272 + 100\angle -6.87^\circ$

$= 0.67204 - j0.10272 + 100\left(\cos\left(-6.87\frac{\pi}{180}\right) + j\sin\left(-6.87\frac{\pi}{180}\right)\right) = 99.954 - j12.$
$064 A$

and the primary phase voltage is

$\widehat{V}_{a_1n_1} = \widehat{E}_{A_1} + \widehat{I}_{A_1}(R_{A_1} + jX_{A_1}) = 125.$
$7\angle 30.92^\circ + (99.954 - j12.064)(0.0395 + j0.0615)$

$= \left(125.7\cos\left(30.92\frac{\pi}{180}\right) + j125.7\sin\left(30.92\frac{\pi}{180}\right)\right) + (99.954 - j12.064)(0.0395 + j0.0615$

$= 112.52 + j70.26 = \sqrt{112.52^2 + 70.26^2}\angle \tan^{-1}\left(\frac{70.26}{112.52}\right)\frac{180}{\pi}$

$= 132.65\angle 31.982^\circ V$

The line voltage on the primary side is

$\widehat{V}_{1L} = \sqrt{3}\,\widehat{V}_{a_1n_1}\angle 30^\circ = 132.65\sqrt{3}\angle 61.982^\circ = 229.76\angle 61.982^\circ V$

The total input power is

$$P_{in} = 3\,\mathrm{Re}\left(\hat{V}_{a_1n_1}\hat{I}_{A_1}^*\right) = 3\,\mathrm{Re}((112.52 + j70.26)(99.954 + j12.064))$$
$$= 3\,\mathrm{Re}(10399 + j8380.2) = 3 \times 10399 = 31197W$$
The total output power is
$$P_o = 3\,\mathrm{Re}\left(\hat{V}_{a_2n_2}\hat{I}_{A_2}^*\right) = 3\,\mathrm{Re}((138.56\angle 0^\circ)(86.6\angle -36.87^\circ))$$
$$= 3\,\mathrm{Re}(138.56 \times 86.6(0.8 + j0.6))$$
$$= 3\,\mathrm{Re}(9599.4 + j7199.6) = 3 \times 9599.4 = 28798W$$
Hence, the efficiency of the three-phase transformer is
$$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{28798}{31197} \times 100 = 92.3\%$$

Chapter 5 Synchronous Machines

The basic components of a synchronous machine are the stator and rotor. The field winding is placed on the rotor and so called the rotor winding. A DC source is connected to the field winding through slip rings and generates a magnetic flux in the machine. Three-phase armature windings are mounted on the stator spatially displaced by 120 degree electrical from one another and so referred to as the stator windings. A three-phase AC source is applied to the armature winding in a synchronous motor to drive a mechanical load and a three-phase AC power is output from the armature winding in a synchronous generator when it is driven by a prime mover.

Two type of rotors are used in the design of synchronous machines, the cylindrical rotor and a salient-pole rotor.

Balanced three-phase currents generate a rotating magnetic field with a constant magnitude and revolving around the periphery of the rotor at a synchronous speed defined by
$$\omega_s = \frac{4\pi f}{P}$$
where f is the frequency of the AC currents and P is the number of poles in the machine.

The induced voltage of a synchronous machine is directly proportional to the product of the flux ϕ in the machine and the speed ω_s of the machine, that is, $E_a = k\phi\omega_s$.

5.1 Synchronous Generators

A generator is driven by a mechanical source, a prime mover, to turn at the synchronous speed. When a DC source is applied to the field winding, three-phase AC voltages are induced in the armature windings.

5.1.1 Synchronous Generators with a Cylindrical Rotor

Figure 5.1 shows an equivalent circuit for a synchronous generator with a cylindrical rotor, where R_f, L_f, V_f, and I_f are the field resistance, inductance, voltage, and current, X_s is the synchronous reactance, R_a, $\hat{V}_a$, and $\hat{I}_a$ are the armature resistance, voltage, and current, $\hat{E}_a$ is the generated voltage (induced voltage). It follows from KVL that
$$\hat{E}_a = \hat{V}_a + \hat{I}_aR_a + j\hat{I}_aX_s$$

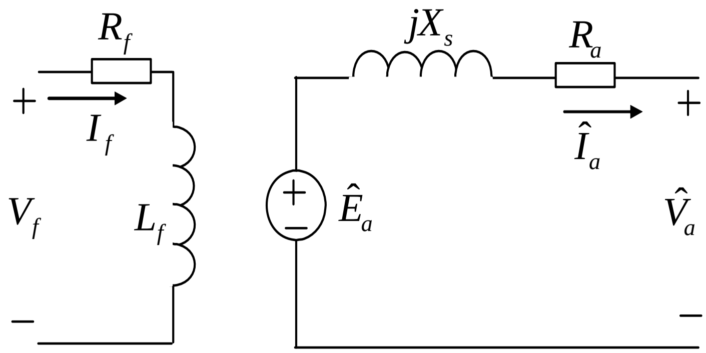


Figure 5.1 The per-phase equivalent circuit of a

Figure 5.2 shows the phasor diagram for a synchronous generator with a lagging load. θ is the power angle and δ is the torque angle. The torque angle is negative for the synchronous generator.

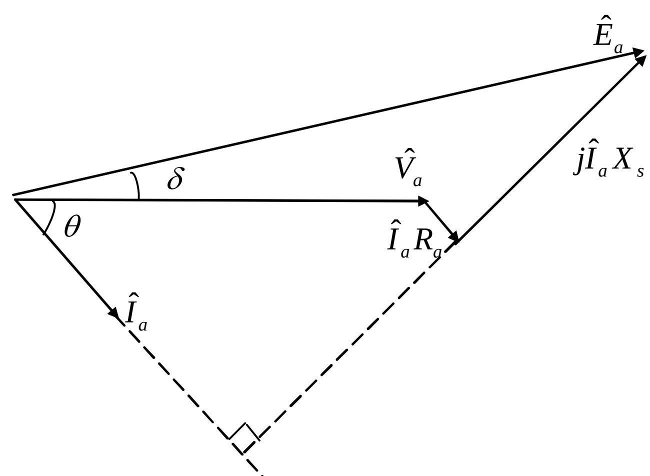


Figure 5.2 The phasor diagram for a

1. **The resistance Test**: This test is conducted to measure the armature winding resistance of a synchronous generator by measuring the resistance from line to line, R_L, when it is at rest and the field winding is open. If the generator is Y-connected, the per-phase resistance is $R_a = 0.5R_L$. On the other hand, for a $\triangle$-connected generator, $R_a = 1.5R_L$.

2. **The Open-Circuit Test** (**No-Load Test**): This test is performed by driving the generator at its rated speed while the armature winding is left open. The open-circuit voltage between any two pair of terminals of the armature windings is recorded when the field current is varied from zero to its rated value. The graph of the per-phase open-circuit voltage versus the field current is referred to as the open-circuit characteristic of a generator.

3. **The Short-Circuit Test**: This tested is carried out by driving the generator at its rated speed when the terminals of the armature winding are shorted. The line current of the armature winding is recorded when the field current is gradually increased. The

graph of the per-phase short-circuit current versus the field current is called the short-circuit characteristic of a generator.

4. **Calculation of the Synchronous Reactance**: Let I_{fr} be the field current which gives the rated per-phase voltage (V_{aoc}) from the open-circuit test and I_{asc} be the phase current corresponding to the field current I_{fr} from the short-circuit test. Then, the synchronous reactance is calculated by

$|Z_s| = \frac{V_{aoc}}{I_{asc}}$ (Synchronous impedance $Z_s = R_a + jX_s$)

$X_s = \sqrt{|Z_s|^2 - R_a^2}$

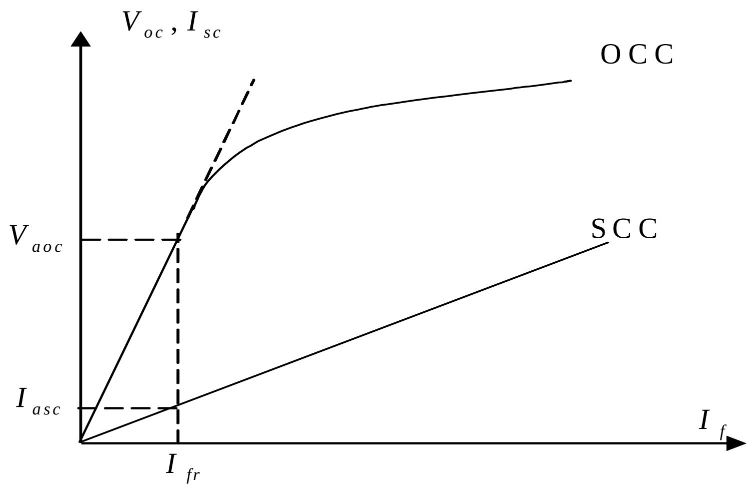


Figure 5.3 OCC and SCC of a synchronous machine

5. **The Developed Torque and Efficiency**:

The output power of a synchronous generator is

$P_o = 3V_aI_a\cos\theta$

The copper loss in the armature winding is

$P_{cu} = 3I_a^2R_a$

The developed power is

$P_d = P_o + P_{cu} = 3V_aI_a\cos\theta + 3I_a^2R_a$

If $R_a = 0$, then

$P_d = \frac{3V_aE_a\sin\delta}{X_s}$

The developed torque is

$\tau_d = \frac{P_d}{\omega_s}$

The input power to the field winding is

$P_f = V_fI_f$

If P_r is the rotational loss and P_{stray} is the stray loss, then the input power $P_{in}(= P_{mech} + P_f)$ is

$P_{in} = P_o + P_{cu} + P_r + P_{stray} + P_f = 3V_aI_a\cos\theta + 3I_a^2R_a + P_r + P_{stray} + P_f$

The core loss $P_c = P_r + P_{stray} + P_f$ does not change much with the load change and can be considered as constant. The efficiency of the generator

$$\eta = \frac{P_o}{P_{in}} = \frac{3V_aI_a\cos\theta}{3V_aI_a\cos\theta + 3I_a^2R_a + P_c}$$

reaches its maximum when

$$3I_a^2R_a = P_c$$

As a result, the generator reaches its maximum efficiency when the load current is

$$I_a = \sqrt{\frac{P_c}{3R_a}}$$

6. **The Voltage Regulation**: The voltage regulation of a synchronous generator is defined as the ratio of the change in the terminal voltage from no load to full load, that is,

$$VR\% = \frac{V_{aNL} - V_{aFL}}{V_{aFL}}$$

where V_{aNL} and V_{aFL} are the no-load voltage and the full-load voltage of the synchronous generator.

Example 5.1: A 500kVA, 2300V, three-phase, Y-connected, synchronous generator is operated at its rated speed to obtain its rated no-load voltage. When a short-circuit is established, the phase current is 150A. The average resistance of each phase is 0.5Ω. The core loss is assumed to be 20kW. Determined the synchronous reactance per phase. Calculate the efficiency and voltage regulation when the generator delivers the rated load at its rated voltage and 0.8 pf lagging.

Solution:

The open-circuit phase voltage is $V_{aoc} = 2300/\sqrt{3} = 1327.9V$

The short-circuit phase current is $I_{asc} = 150A$

Therefore the synchronous impedance is $|Z_s| = \frac{V_{aoc}}{I_{asc}} = \frac{1327.9}{150} = 8.85\Omega$

Thus the synchronous reactance is $X_s = \sqrt{|Z_s|^2 - R_a^2} = \sqrt{8.85^2 - 0.5^2} = 8.84\Omega$

The rated phase voltage is $V_a = \frac{2300}{\sqrt{3}} = 1327.9V$ and it is assumed that $\hat{V}_a = 1327.9\angle 0^\circ V$

The rated load current is $I_a = \frac{500000}{3\times 1327.9} = 125.51A$ and $\hat{I}_a = 125.51\angle -36.87^\circ A$

It follows from the per-phase equivalent circuit that

$$\hat{E}_a = \hat{V}_a + \hat{I}_aZ_s = 1327.9\angle 0^\circ + 125.51\angle -36.87^\circ(0.5 + j8.84)$$
$$= 1327.9 + 125.51(0.8 - j0.6)(0.5 + j8.84) = 2043.8 + j849.95$$
$$= \sqrt{2043.8^2 + 849.95^2}\angle \tan^{-1}\left(\frac{849.95}{2043.8}\right)\frac{180}{\pi} = 2213.5\angle 22.6^\circ V$$

Thus, the no-load phase voltage is $V_{aNL} = E_a = 2213.5V$ and the full-load phase voltage is $V_{aFL} = V_a = 1327.9V$, which implies that

$$VR\% = \frac{2213.5 - 1327.9}{1327.9} \times 100 = 66.7\%$$

The output power of the synchronous generator is

$$P_o = 3V_aI_a\cos\theta = 3 \times 1327.9 \times 125.51 \times 0.8 = 400000W$$

The copper loss in the armature winding is

$$P_{cu} = 3I_a^2R_a = 3 \times 125.51^2 \times 0.5 = 23629W$$

The input power is

$$P_{in} = P_o + P_{cu} + P_c = 400000 + 23629 + 20000 = 443630W$$

Thus, the efficiency of the generator is

$$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{400000}{443630} \times 100 = 90.2\%$$

5.1.2 Synchronous Generators with a Salient-Pole Rotor

Unlike a cylindrical rotor synchronous generator, a salient-pole synchronous generator has a large air-gap in the region between the poles than in the region just above the poles, as is evidenced from Figure 5.3. Therefore, the reluctances of the two regions in a salient-pole generator differ significantly.

In order to account for this difference, the synchronous reactance is split into two reactances. The component of the synchronous reactance along the pole-axis (the d-axis) is called the direct-axis synchronous reactance X_d and the other component along the axis between the poles (the q-axis) is referred to as the quadrature-axis synchronous reactance X_q.

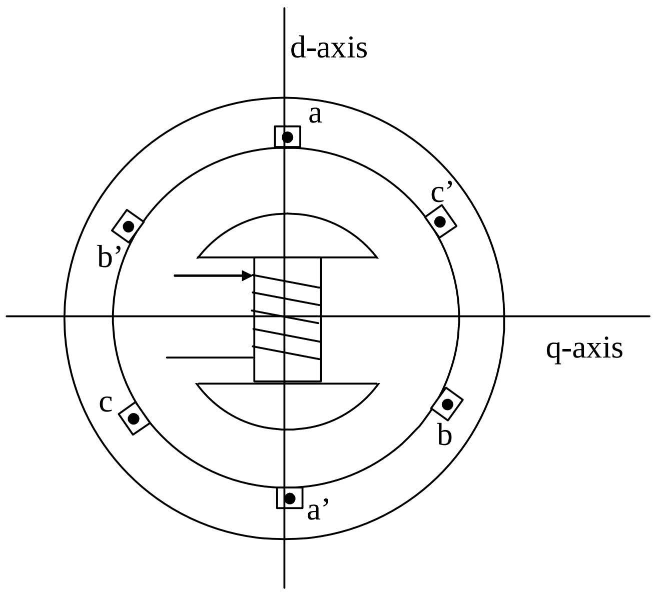


Figure 5.4 A salient-pole synchronous

The armature current $\hat{I}_a$ is also resolved into two components: the direct-axis component $\hat{I}_d$ and quadrature-axis component $\hat{I}_q$. Then, $\hat{I}_a = \hat{I}_d + \hat{I}_q$. The direct-axis component $\hat{I}_d$ produces the field along the d-axis and lags $\hat{E}_a$ by 90° and the quadrature-axis component $\hat{I}_q$ produces the field along the q-axis and is in phase with $\hat{E}_a$.

Let $\hat{E}_a$ be the per-phase generated voltage under no-load and $\hat{E}_d$ and $\hat{E}_q$ be the induced emfs in the armature winding by the currents $\hat{I}_d$ and $\hat{I}_q$, respectively. Then $\hat{E}_d$ and $\hat{E}_q$ can be expressed in terms of X_d and X_q as

$\hat{E}_d = -j\hat{I}_d X_d$ and $\hat{E}_q = -j\hat{I}_q X_q$

The per-phase terminal voltage of the generator is

$$\hat{V}_a = \hat{E}_a + \hat{E}_d + \hat{E}_q - \hat{I}_a R_a = \hat{E}_a - j\hat{I}_d X_d - j\hat{I}_q X_q - \hat{I}_a R_a$$

$$= \hat{E}_a - j\hat{I}_d X_d - j\left(\hat{I}_a - \hat{I}_d\right)X_q - \hat{I}_a R_a = \hat{E}_a - j\hat{I}_d(X_d - X_q) - j\hat{I}_a X_q - \hat{I}_a R_a$$

$$= \hat{E}_a' - j\hat{I}_a X_q - \hat{I}_a R_a$$

where $\hat{E}_a' = \hat{E}_a - j\hat{I}_d(X_d - X_q)$, as shown in Figure 5.4.

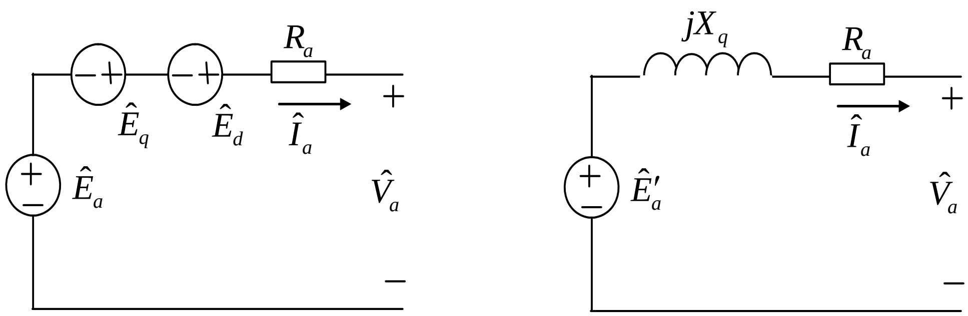
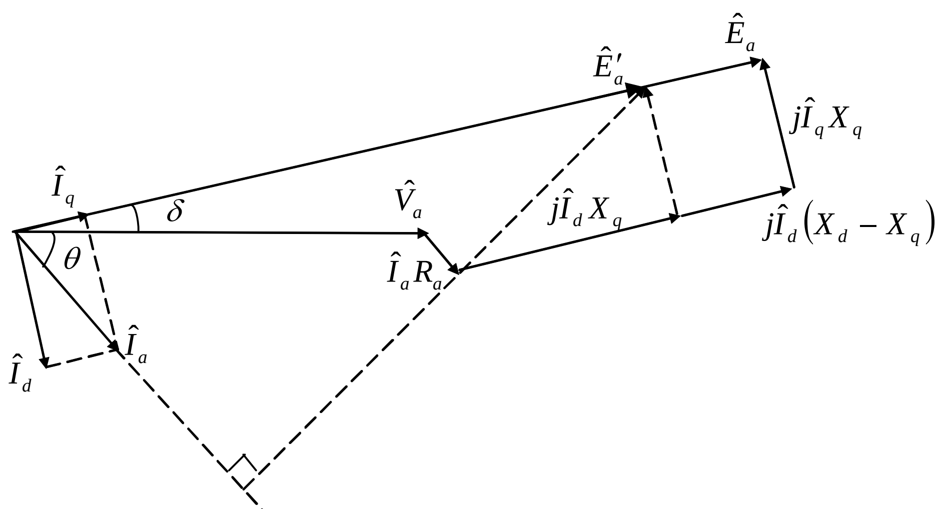


Figure 5.5 Equivalent circuits of a salient-pole synchronous

The phasor diagrams for a lagging load and a leading load are shown in Figure 5.5.



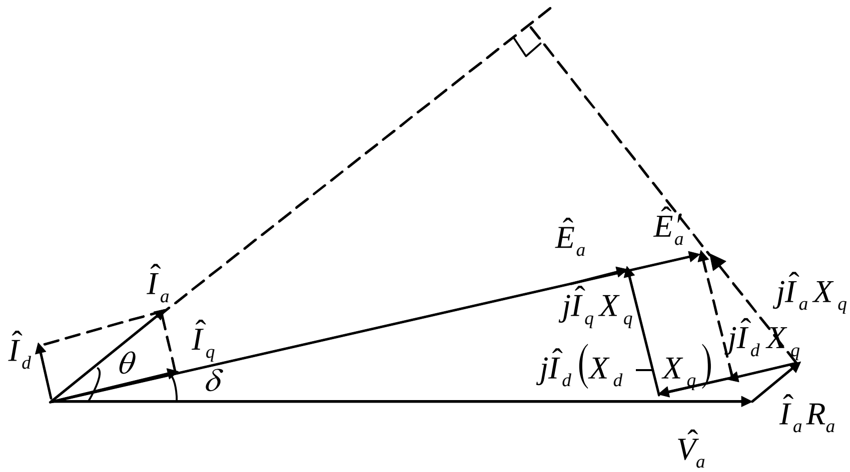


Figure 5.6 Phasor diagram of a salient-pole synchronous

The output power is

$$P_o = 3\,\mathrm{Re}\left(\hat{V}_a \hat{I}_a^*\right) = 3V_a I_a \cos\theta$$

The copper loss in the armature winding is

$$P_{cu} = 3I_a^2 R_a$$

The developed power is

$$P_d = P_o + P_{cu} = 3V_a I_a \cos\theta + 3I_a^2 R_a$$

If $R_a = 0$, then

$$P_d = \frac{3V_a E_a \sin|\delta|}{X_d} + \frac{3(X_d - X_q)}{2X_d X_q} V_a^2 \sin|2\delta|$$

and the developed torque is

$$\tau_d = \frac{3V_a E_a \sin|\delta|}{X_d \omega_s} + \frac{3(X_d - X_q)}{2X_d X_q \omega_s} V_a^2 \sin|2\delta|$$

The input power to the field winding is

$$P_f = V_f I_f$$

If P_r is the rotational loss and P_{stray} is the stray loss, then the input power $P_{in}(= P_{mech} + P_f)$ is

$$P_{in} = P_o + P_{cu} + P_r + P_{stray} + P_f = 3V_a I_a \cos\theta + 3I_a^2 R_a + P_r + P_{stray} + P_f = 3V_a I_a \cos\theta + 3I_a^2 R_a$$

where $P_c = P_r + P_{stray} + P_f$ is the core loss. The efficiency of the generator

$$\eta = \frac{P_o}{P_{in}} = \frac{3V_a I_a \cos\theta}{3V_a I_a \cos\theta + 3I_a^2 R_a + P_c}$$

Example 5.2: A 70MVA 13.8kV 60Hz two-pole Y-connected three-phase salient-pole synchronous generator has $R_a = 0$, $X_d = 1.83\Omega$, and $X_q = 1.21\Omega$. It delivers the rated load at 0.8 pf lagging. Determine δ, $\hat{E}_a$, $VR\%$, P_d, and τ_d.

Solution: The phase terminal voltage is

$$\hat{V}_a = \frac{13800}{\sqrt{3}} \angle 0^\circ = 7967.4\angle 0^\circ V$$

The phase load current is

$$\hat{I}_a = \frac{70\times 10^6}{\sqrt{3}\times 13800} \angle -36.87^\circ = 2928.6\angle -36.87^\circ A$$

It follows from the equivalent circuit that

$$\widehat{E}'_a = \widehat{V}_a + j\widehat{I}_aX_q + \widehat{I}_aR_a = 7967.4\angle 0^\circ + (2928.6\angle -36.87^\circ)(j1.21)$$

$$= 7967.4 + 2928.6(0.8 - j0.6)(j1.21) = 10094 + j2834.9$$

$$= \sqrt{10094^2 + 2834.9^2}\angle \tan^{-1}\left(\tfrac{2834.9}{10094}\right)\tfrac{180}{\pi} = 10485\angle 15.7^\circ V$$

The torque angle is 15.7°. The d- and q-axis currents are

$$\widehat{I}_d = I_a\sin(|\theta| + \delta)\angle(-90^\circ + \delta)$$

$$= 2928.6\sin\left((36.87 + 15.7)\tfrac{\pi}{180}\right)\angle(-90^\circ + 15.7^\circ) = 2323.6\angle -74.3^\circ A$$

$$\widehat{I}_q = I_a\cos(|\theta| + \delta)\angle\delta = 2928.6\cos\left((36.87 + 15.7)\tfrac{\pi}{180}\right)\angle 15.7^\circ = 1780\angle 15.7^\circ A$$

The generated voltage is

$$\widehat{E}_a = \widehat{E}'_a + j\widehat{I}_d(X_d - X_q) = 10094 + j2834.9 + (2323.6\angle -74.3^\circ)(j(1.83 - 1.21))$$

$$= 10094 + j2834.9 + 2323.6\left(\cos\left((-74.3)\tfrac{\pi}{180}\right) + j\sin\left((-74.3)\tfrac{\pi}{180}\right)\right)(j(1.83 - 1.21))$$

$$= 11481 + j3224.7 = \sqrt{11481^2 + 3224.7^2}\angle \tan^{-1}\left(\tfrac{3224.7}{11481}\right)\tfrac{180}{\pi} = 11925\angle 15.7^\circ V$$

or is given by

$$\widehat{E}_a = \widehat{V}_a + j\widehat{I}_dX_d + j\widehat{I}_qX_q + \widehat{I}_aR_a$$

$$= 7967.4\angle 0^\circ + (2323.6\angle -74.3^\circ)(j1.83) + (1780\angle 15.7^\circ)(j1.21) + 0$$

$$= 7967.4 + 2323.6\left(\cos\left((-74.3)\tfrac{\pi}{180}\right) + j\sin\left((-74.3)\tfrac{\pi}{180}\right)\right)(j1.83)$$

$$+ 1780\left(\cos\left((15.7)\tfrac{\pi}{180}\right) + j\sin\left((15.7)\tfrac{\pi}{180}\right)\right)(j1.21)$$

$$= 11478 + j3224.1 = \sqrt{11478^2 + 3224.1^2}\angle \tan^{-1}\left(\tfrac{3224.1}{11478}\right)\tfrac{180}{\pi} = 11922\angle 15.7^\circ V$$

The developed power is

$$P_d = P_o + P_{cu} = P_o = 3V_aI_a\cos\theta = 3 \times 7967.4 \times 2928.6 \times 0.8 = 5.6 \times 10^7 W$$

The synchronous speed is

$$\omega_s = \tfrac{4\pi f}{P} = \tfrac{4\pi\times 60}{2} = 376.99 rad/s$$

The developed torque is

$$\tau_d = \tfrac{P_d}{\omega_s} = \tfrac{5.6\times 10^7}{376.99} = 1.4855 \times 10^5 N\bullet m$$

The voltage regulation is

$$VR\% = \tfrac{11925 - 7967.4}{7967.4} \times 100 = 49.7\%$$

5.2 Synchronous Motors

A synchronous motor is powered by a electrical source to drive a load at the synchronous speed. When a DC source is applied to the field winding and three-phase AC voltages are connected to the armature windings, the motor will turns its load at the synchronous speed.

5.2.1 Synchronous Motors with a Cylindrical Rotor

The equivalent circuit for a cylindrical-rotor synchronous motor is shown in Figure 5.7, which is the same as the cylindrical-rotor synchronous generator with the reversed armature current direction. It follows from Kirchhoff’s voltage law that

$$\widehat{V}_a = \widehat{E}_a + \widehat{I}_aR_a + j\widehat{I}_aX_s$$

Figure 5.8 shows the phasor diagrams for a synchronous motor with a lagging load. θ is the power angle and δ is the torque angle. The torque angle is negative for the synchronous motor.

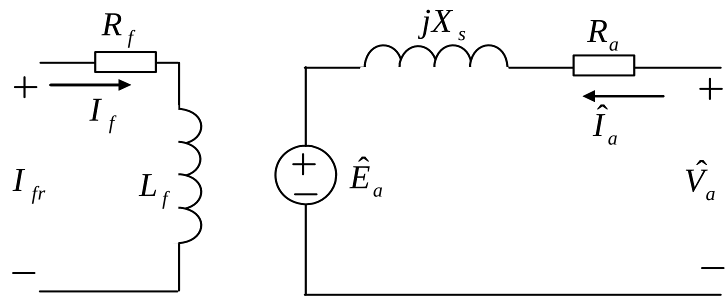


Figure 5.7 The per-phase equivalent circiut of a

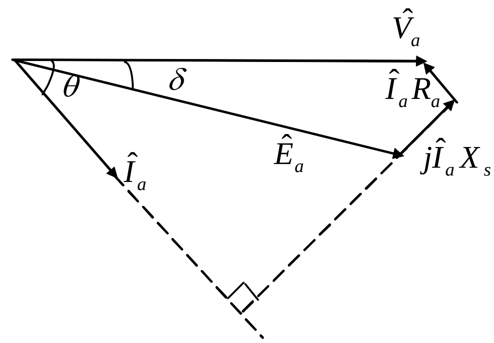


Figure 5.8 The phasor diagram

The input power of a synchronous motor is

$$P_{in} = 3V_a I_a \cos\theta + V_f I_f$$

The copper loss is

$$P_{cu} = 3I_a^2 R_a + V_f I_f$$

The developed power is

$$P_d = P_{in} - P_{cu} = 3V_a I_a \cos\theta - 3I_a^2 R_a$$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_s}$$

If $R_a = 0$, then

$$P_d = \frac{3V_a E_a \sin\delta}{X_s}$$

If P_r is the rotational loss and P_{stray} is the stray loss, then the output power $P_o (= \tau_o \omega_s)$ is

$$P_o = P_d - P_r - P_{stray}$$

The efficiency of the motor is

$$\eta = \frac{P_o}{P_{in}}$$

Example 5.3: A 220V 60Hz 3-phase 2-pole Y-connected synchronous motor has a synchronous impedance of 0.25+j2.5Ω/phase. The motor delivers the rated load of 80A at 0.707 pf leading. Determine (a) the generated voltage, (b) the torque angle, (c) the power developed by the motor, and (d) the developed torque.

Solution: The phase voltage is $V_a = \frac{220}{\sqrt{3}} = 127V$. Assume $\widehat{V}_a = 127\angle 0^\circ V$. The phase armature current is $\hat{I}_a = 80\angle 45^\circ A$. It follows from the per-phase equivalent circuit that

$$\hat{E}_a = \hat{V}_a - \hat{I}_a R_a - j\hat{I}_a X_s = 127\angle 0^\circ - (80\angle 45^\circ)(0.25 + j2.5)$$
$$= 127 - 80(0.701 + j0.707)(0.25 + j2.5) = 254.38 - j154.34$$
$$= \sqrt{254.38^2 + 154.34^2}\angle \tan^{-1}\left(\frac{-154.34}{254.38}\right)\frac{180}{\pi} = 297.54\angle -31.2^\circ V$$

The torque angle is -31.2°.

$P_d = P_{in} - P_{cu} = 3V_a I_a \cos\theta + P_f - (3I_a^2 R_a + P_f) = 3 \times (127 \times 80 \times 0.707 - 80^2 \times 0.25) = 16749W$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_s} = \frac{16749}{\frac{4\pi \times 60}{2}} = 44.428 N \cdot m$$

5.2.2 Synchronous Motors with a Salient-Pole Rotor

Similar to a salient-pole synchronous generator, the per-phase equivalent circiut for a salient-pole synchronous motor is required to analyse the motor performance, which is shown in Figure 5.9.

The per-phase terminal voltage of the motor is

$$\hat{V}_a = \hat{E}_a - \hat{E}_d - \hat{E}_q + \hat{I}_a R_a = \hat{E}_a + j\hat{I}_d X_d + j\hat{I}_q X_q + \hat{I}_a R_a$$
$$= \hat{E}_a + j\hat{I}_d X_d + j\left(\hat{I}_a - \hat{I}_d\right) X_q + \hat{I}_a R_a = \hat{E}_a + j\hat{I}_d (X_d - X_q) + j\hat{I}_a X_q + \hat{I}_a R_a$$
$$= \hat{E}_a' + j\hat{I}_a X_q + \hat{I}_a R_a$$

where $\hat{E}_a' = \hat{E}_a + j\hat{I}_d (X_d - X_q)$, as shown in Figure 5.9. The phasor diagrams for a leading load and a lagging load are shown in Figure 5.10.

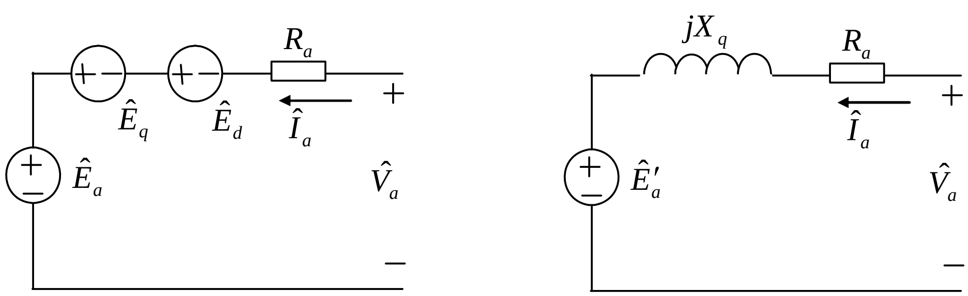


Figure 5.9 The equivalent circiut of a salient-pole motor

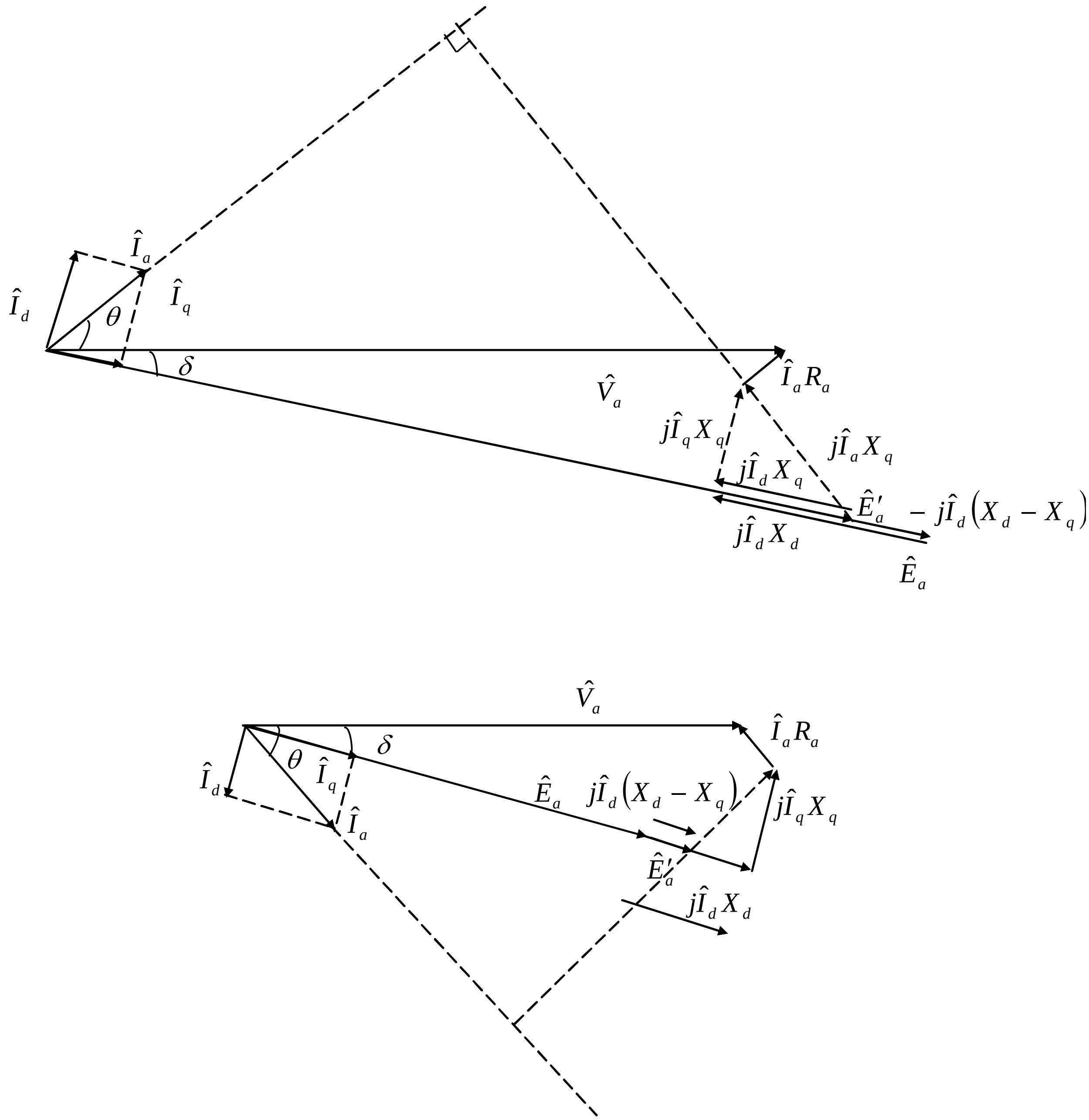


Figure 5.10 The Phasor diagram

Then the input power is

$$P_{in} = 3\,\mathrm{Re}\left(\hat{V}_a \hat{I}_a^*\right) = 3V_a I_a \cos\theta + V_f I_f$$

The copper loss is

$$P_{cu} = 3I_a^2 R_a + V_f I_f$$

The developed power is

$$P_d = P_{in} - P_{cu} = 3V_a I_a \cos\theta - 3I_a^2 R_a$$

If $R_a = 0,$ then

$$P_d = \frac{3V_aE_a\sin|\delta|}{X_d} + \frac{3(X_d - X_q)}{2X_dX_q}V_a^2\sin|2\delta|$$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_s}$$

If P_r is the rotational loss and P_{stray} is the stray loss, then the output power $P_o(= \tau_o\omega_s)$ is

$$P_o = P_d - P_r - P_{stray}$$

The efficiency of the motor

$$\eta = \frac{P_o}{P_{in}}$$

Example 5.4: A 208V 60Hz three-phase Y-connected salient-pole synchronous motor operates at full load and draws a current of 40A at 0.8 pf lagging. The d- and q-axis reactances are 2.7$\Omega/phase$ and 1.7$\Omega/phase$, respectively. The armature-winding resistance is negligible, and the rotational loss is 5% of the power developed by the motor. Determine (a) the developed voltage, (b) the developed power, and (c) the efficiency.

Solution: The per-phase load voltage and current are $\hat{V}_a = \frac{208}{\sqrt{3}}\angle 0^\circ = 120\angle 0^\circ V$ and $\hat{I}_a = 40\angle - \cos^{-1}(0.8) = 40\angle - 36.87^\circ A$.

It follows from the per-phase equivalent circuit that

$$\hat{E}_a' = \hat{V}_a - j\hat{I}_aX_q - \hat{I}_aR_a =$$

$$120\angle 0^\circ - (40\angle - 36.87^\circ)(j1.7) = 120 - 40(0.8 - j0.6)(j1.7)$$

$$= 79.2 - j54.4 = \sqrt{79.2^2 + 54.4^2}\angle\tan^{-1}\left(\frac{-54.4}{79.2}\right)\frac{180}{\pi} = 96.083\angle - 34.48^\circ V$$

which means that the torque angle is $\delta = -34.48^\circ$. It follows from the phasor diagram that the absolute value of the angle between $\hat{I}_q$ and $\hat{I}_a$ are $\alpha = |\theta - \delta| = |-36.87 - (-34.48)| = 2.39^\circ$. Therefore, the d-axis armature current is

$$\hat{I}_d = I_a\sin\alpha\angle(\delta - 90^\circ) = 40\sin\left(2.39\frac{\pi}{180}\right)\angle(-34.48 - 90)^\circ = 1.668\angle - 124.48^\circ$$

$$= 1.668\left(\cos\left(-124.48\frac{\pi}{180}\right) + j\sin\left(-124.48\frac{\pi}{180}\right)\right) = -0.944 - j1.375A$$

(a) The per-phase developed voltage is

$$\hat{E}_a = \hat{E}_a' - j\hat{I}_d(X_d - X_q) = (79.2 - j54.4) - j(-0.944 - j1.375)(2.7 - 1.7)$$

$$= 77.825 - j53.456 = \sqrt{77.825^2 + 53.456^2}\angle\tan^{-1}\left(\frac{-53.456}{77.825}\right)\frac{180}{\pi} = 94.415\angle - 34.48^\circ V$$

(b) As $R_a = 0$, the AC input power is the same as the developed power, that is,

$$P_d = P_{in} - P_{cu} = P_{in} = 3V_aI_a\cos\theta = 3\times 120\times 40\times 0.8 = 11520W$$

Or it can be calculated by

$$P_d = \frac{3V_aE_a\sin|\delta|}{X_d} + \frac{3(X_d - X_q)}{2X_dX_q}V_a^2\sin|2\delta|$$

$$= \frac{3\times 120\times 94.415\times\sin\left(34.48\frac{\pi}{180}\right)}{2.7} + \frac{3(2.7 - 1.7)}{2\times 2.7\times 1.7}\times 120^2\times\sin\left(2\times 34.48\frac{\pi}{180}\right) = 11519W$$

(c) The output power is

$$P_o = P_{in} - P_{cu} - P_r - P_{stray} = P_{in} - P_r = 11520 - 0.05\times 11520 = 10944W$$

and the efficiency is

$$\eta = \frac{P_o}{P_{in}}\times 100 = \frac{10944}{11520}\times 100 = 95\%$$

5.2.3 Power Factor Correction

Example 5.5: A manufaturing plant uses 100kVA at 0.6 pf lagging under normal operation. Select a synchronous AC motor to improve the overall power factor to 0.9 lagging.

Solution: $S_L = 100kVA$, $\theta_L = \cos^{-1}(0.6) * \frac{180}{\pi} = 53.13^\circ$. Thus,

$$P_L = S_L \cos\theta_L = 60kW$$

$$Q_L = S_L \sin\theta_L = 100\sin\left(53.13 \times \frac{\pi}{180}\right) = 80kVAR$$

Assume the synchronous motor rating is S_m with power angle θ_m. Then,

$$P_m = S_m \cos\theta_m$$

$$Q_m = S_m \sin\theta_m$$

The total powers are

$$P = P_L + P_m = P_L + S_m \cos\theta_m$$

$$Q = Q_L + Q_m = Q_L + S_m \sin\theta_m$$

The overall power angle is $\theta = \cos^{-1}(0.9) * \frac{180}{\pi} = 25.84^\circ$, i.e. $\tan\theta = \tan\left(25.84 \times \frac{\pi}{180}\right) = 0.48$. So,

$$\tan\theta = \frac{Q}{P} = \frac{Q_L + S_m \sin\theta_m}{P_L + S_m \cos\theta_m}$$

that is,

$$P_L \tan\theta + S_m \tan\theta \cos\theta_m = Q_L + S_m \sin\theta_m$$

or

$$P_L \tan\theta + S_m \tan\theta \cos\theta_m = Q_L + S_m\sqrt{1-\cos^2\theta_m} \Rightarrow (P_L \tan\theta - Q_L) + S_m \tan\theta \cos\theta_m = S_m\sqrt{1-\cos^2\theta_m}$$

Squaring both sides yields

$$(P_L \tan\theta - Q_L)^2 - S_m^2 + 2(P_L \tan\theta - Q_L)S_m \tan\theta \cos\theta_m + \left[(S_m \tan\theta)^2 + S_m^2\right]\cos^2\theta_m = 0$$

Solving this equation for $\cos\theta_m$ gives

$$\cos\theta_m = \frac{-2(P_L\tan\theta - Q_L)S_m\tan\theta \pm \sqrt{4[(P_L\tan\theta-Q_L)S_m\tan\theta]^2 - 4\left[(S_m\tan\theta)^2+S_m^2\right]\left[(P_L\tan\theta-Q_L)^2-S_m^2\right]}}{2\left[(S_m\tan\theta)^2+S_m^2\right]}$$

$$= \frac{-(P_L\tan\theta-Q_L)\tan\theta \pm \sqrt{[(P_L\tan\theta-Q_L)\tan\theta]^2 - [\tan^2\theta+1]\left[(P_L\tan\theta-Q_L)^2-S_m^2\right]}}{S_m(\tan^2\theta+1)}$$

$$= \frac{-(P_L\tan\theta-Q_L)\tan\theta \pm \sqrt{(\tan^2\theta+1)S_m^2 - (P_L\tan\theta-Q_L)^2}}{S_m(\tan^2\theta+1)}$$

It is necessary to have

$(\tan^2\theta + 1)S_m^2 - (P_L\tan\theta - Q_L)^2 \geq 0$ and

$$-(P_L\tan\theta - Q_L)\tan\theta - \sqrt{(\tan^2\theta+1)S_m^2 - (P_L\tan\theta-Q_L)^2} \geq 0$$

that is,

$$S_m \geq \sqrt{\frac{(P_L\tan\theta - Q_L)^2}{(\tan^2\theta+1)}} = \sqrt{\frac{(60\times0.48-80)^2}{(0.48^2+1)}} = 47kVA$$

and

$$\sqrt{(\tan^2\theta+1)S_m^2 - (P_L\tan\theta-Q_L)^2} \leq (P_L\tan\theta - Q_L)\tan\theta$$

$$S_m \leq -P_L\tan\theta + Q_L = -60 \times 0.48 + 80 = 51.2$$

For different synchronous motor ratings S_m, one can get different power factors. Now choose $S_m = 47kVA$, then

$$\cos\theta_m = \frac{-(P_L\tan\theta - Q_L)\tan\theta \pm \sqrt{(\tan^2\theta+1)S_m^2-(P_L\tan\theta-Q_L)^2}}{S_m(\tan^2\theta+1)}$$

$$= \frac{-(60\times0.48-80)\times0.48\pm\sqrt{(0.48^2+1)\times47^2-(60\times0.48-80)^2}}{47\times(0.48^2+1)} = 0.59, 0.26$$

Therefore, $\theta_m = -\cos^{-1}(0.59) * \frac{180}{\pi} = -53.8^\circ, \theta_m = -\cos^{-1}(0.26) * \frac{180}{\pi} = -74.9^\circ$, which gives

$$P = 60 + 47 \times 0.59 = 87.7kW$$
$$Q = 80 + 47 \times \sin\left(-53.8 \times \frac{\pi}{180}\right) = 42.0kVAR$$

or

$$P = 60 + 47 \times 0.26 = 72.2kW$$
$$Q = 80 + 47 \times \sin\left(-74.9 \times \frac{\pi}{180}\right) = 34kVAR$$

Selecting $S_m = 50kVA$ produces

$$\cos\theta_m = \frac{-(P_L\tan\theta - Q_L)\tan\theta \pm \sqrt{(\tan^2\theta+1)S_m^2-(P_L\tan\theta-Q_L)^2}}{S_m(\tan^2\theta+1)}$$

$$= \frac{-(60\times0.48-80)\times0.48\pm\sqrt{(0.48^2+1)\times50^2-(60\times0.48-80)^2}}{50\times(0.48^2+1)} = 0.75, 0.05$$

Therefore, $\theta_m = -\cos^{-1}(0.75) * \frac{180}{\pi} = -41.4^\circ, \theta_m = -\cos^{-1}(0.05) * \frac{180}{\pi} = -87.1^\circ$, which gives

$$P = 60 + 47 \times 0.75 = 95.3kW$$
$$Q = 80 + 47 \times \sin\left(-41.4 \times \frac{\pi}{180}\right) = 48.9kVAR$$

or

$$P = 60 + 47 \times 0.05 = 62.4kW$$
$$Q = 80 + 47 \times \sin\left(-87.1 \times \frac{\pi}{180}\right) = 33.0kVAR$$

5.3 V-Curves for Synchronous Machines

Chapter 6 Induction Motors

6.1 Three-Phase Induction Motors

The essential components of an induction motor are a stator and a rotor. A balanced three-phase winding is placed on the stator. There are two types of rotors: a squirrel-cage rotor and a wound rotor. Rotor windings are short-circuited for both types of rotors. When the stator winding of a three-phase induction motor is connected to a three-phase power supply, it produces a rotating magnetic field which is constant in magnitude and revolves at the synchronous speed given by

$\omega_s = \frac{4\pi f}{P}$ or $N_s = \frac{120f}{P}$

where f is the frequency of the power supply and P is the number of poles. This rotating magnetic field induces emf in the rotor winding. Since the rotor winding is short-circuited, the induced emf produces an induced current in the rotor winding, which, together with the rotating magnetic field, induces torque on the rotor winding to make the rotor spin at speed ω_m. It is important to note that the induced voltage is proportional to the relative speed of the rotor with respect to the synchronous speed of the rotating magnetic field. Such a relative speed is defined as the slip speed

$\omega_r = \omega_s - \omega_m$ or $N_r = N_s - N_m$

and the ratio between the relative speed and the synchronous speed is referred to as the slip

$s = \frac{\omega_s - \omega_m}{\omega_s} = \frac{N_s - N_m}{N_s}$

The motor speed can be expressed as

$\omega_m = (1 - s)\omega_s$ or $N_m = (1 - s)N_s$

The frequency of the induced voltage in the rotor is

$f_r = \frac{PN_r}{120} = \frac{P(N_s - N_m)}{120} = \frac{PN_s}{120}\frac{N_s - N_m}{N_s} = sf$

When the rotor is stationary, the slip is 1 and the rotor appears exactly like a short-circuited secondary winding of a transformer. Therefore, an induction motor is a transformer with a rotating secondary winding and the equivalent circuit for a transformer can be used for an induction motor.

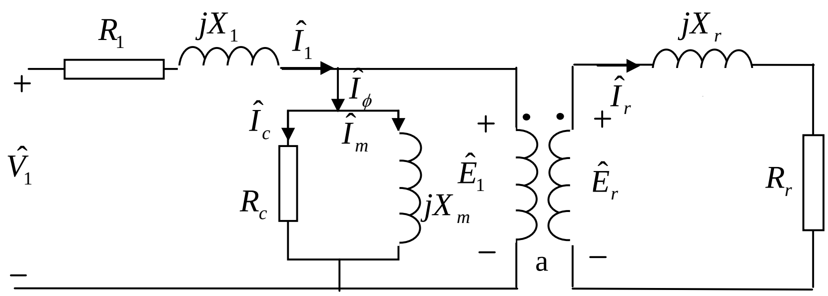


Figure 6.1

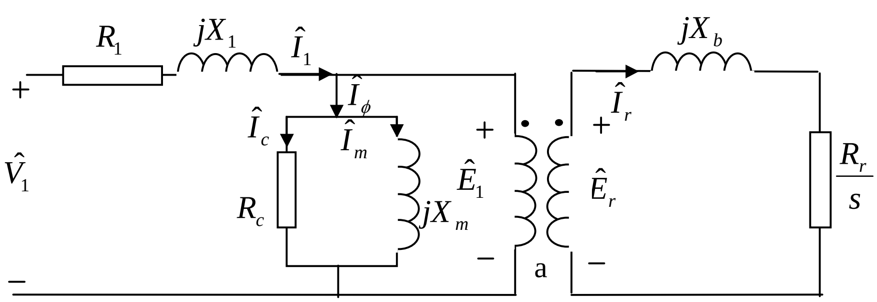


Figure 6.2

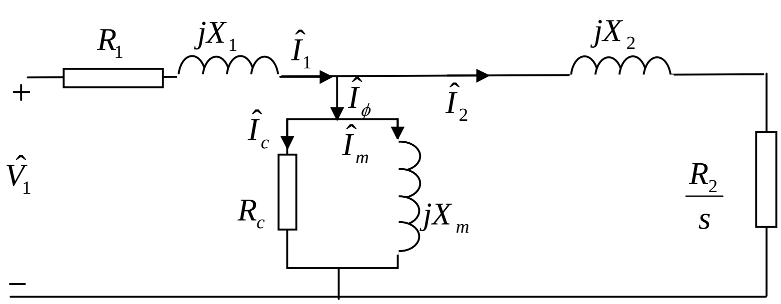


Figure 6.3

Figure 6.1 shows a per-phase equivalent circuit for a three-phase induction motor, where

$\hat{V}_1$ =per-phase stator voltage
$\hat{I}_1$ =per-phase stator current
R_1 =per-phase stator resistance
L_1 =per-phase stator leakage inductance
$X_1 = 2\pi f L_1$ =per-phase stator leakage reactance
$\hat{I}_r$ =per-phase rotor current
R_r =per-phase rotor resistance
L_r =per-phase rotor leakage inductance
$X_b = 2\pi f L_r$ =per-phase rotor leakage reactance at s=1
$X_r = 2\pi f_r L_r = sX_b$ =per-phase rotor leakage reactance
X_m =per-phase magnetization reactance
R_c =per-phase core-loss resistance
$\hat{E}_1$ =per-phase induced voltage in the stator winding
$\hat{E}_b$ =per-phase induced voltage in the rotor winding at s=1
$\hat{E}_r = s\hat{E}_b$ =per-phase induced voltage in the rotor winding
$\hat{I}_\phi = \hat{I}_c + \hat{I}_m$ =per-phase excitation current
$\hat{I}_c$ =per-phase core-loss current
$\hat{I}_m$ =per-phase magnetization current
a =effective turns ratio

Note that

$$\hat{I}_r = \frac{\hat{E}_r}{R_r + jX_r} = \frac{s\hat{E}_b}{R_r + jsX_b} = \frac{\hat{E}_b}{\frac{R_r}{s} + jX_b}$$

Hence the equivalent circuit Figure 6.1 can be modified as Figure 6.2. Referring the rotor side to the stator side, the equivalent circuit Figure 6.2 is transformed to the equivalent circuit Figure 6.3. The approximate per-phase equivalent circuit is given as Figure 6.4.

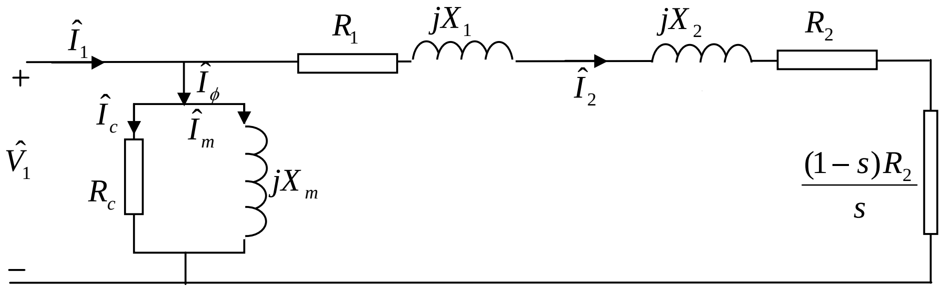


Figure 6.4

The Stator Resistance Test

Let R_L be the DC value of the resistance between any two terminals of the motor. Then the per-phase resistance is

$R_1 = 0.5R_L$ for Y-connection

$R_1 = 1.5R_L$ for △-connection

The Blocked-Rotor Test

The rotor is held still by external torque and a variable three-phase power is applied to the stator winding. The stator voltage is carefully increased from zero until the motor draws the rated current. Let V_{br}, I_{br}, and P_{br} be the input voltage, current, and power on a per-phase basis. Then,

$$R_e = \frac{P_{br}}{I_{br}^2}$$

$$|Z_e| = \frac{V_{br}}{I_{br}^2}$$

where $Z_e = R_e + jX_e = R_1 + R_2 + j(X_1 + X_2)$. Therefore,

$$R_2 = R_e - R_1$$

$$X_e = \sqrt{|Z_e|^2 - R_e^2}$$

For all practical purposes, it is assumed that $X_1 = X_2 = 0.5X_e$

The No-Load Test

The rated voltage is applied to the stator winding and the motor operates without any load. Let V_{NL}, I_{NL}, and P_{NL} be the input voltage, current, and power on a per-phase basis. Let P_r be the rotational loss on a per-phase basis. Then the power loss in R_c is

$$P_c = P_{NL} - P_r$$

and

$$R_c = \frac{V_{NL}^2}{P_c}$$

$$|Z_\phi| = \frac{V_{NL}}{I_{NL}}$$

where $Z_\phi = \frac{1}{\frac{1}{R_c} + \frac{1}{jX_m}}$. Note that $\left|\frac{1}{Z_\phi}\right|^2 = \left(\frac{1}{R_c}\right)^2 + \left(\frac{1}{X_m}\right)^2$. As a result, we have

$$X_m = \frac{1}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2 - \left(\frac{1}{R_c}\right)^2}}$$

Power Flow Diagram

The following are based on the exact per-phase equivalent circuit.

The input power: $P_{in} = 3V_1 I_1 \cos\theta$

The stator copper loss: $P_{scu} = 3I_1^2 R_1$

The air-gap power: $P_{ag} = P_{in} - P_{scu} = 3I_2^2 \frac{R_2}{s}$ (the power consumed by $\frac{R_2}{s}$)

The rotor copper loss: $P_{rcu} = 3I_2^2 R_2 = sP_{ag}$

The developed power: $P_d = P_{ag} - P_{rcu} = P_{ag} - sP_{ag} = (1-s)P_{ag} = 3I_2^2 \frac{(1-s)R_2}{s}$

The rotational loss: $P_r = P_c + P_{fw} + P_{stray}$

The output power: $P_o = P_d - P_r$

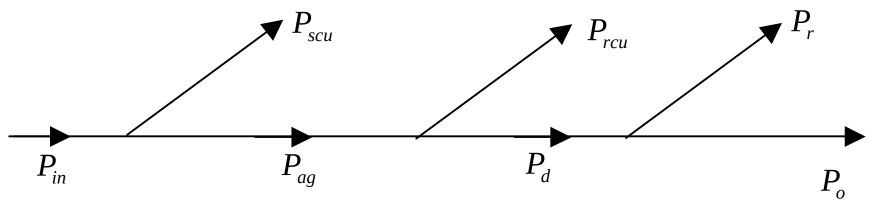


Figure 6.5

The following are based on the approximate equivalent circuit.

It follows from the approximate equivalent circuit Figure 6.4 that the rotor current is

$$\hat{I}_2 = \frac{\hat{V}_1}{R_1+R_2+j(X_1+X_2)+\frac{(1-s)R_2}{s}} = \frac{V_1}{\sqrt{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}}\angle\theta_2$$

So the developed power is

$$P_d = 3I_2^2\frac{(1-s)R_2}{s} = \frac{3V_1^2\frac{(1-s)R_2}{s}}{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}$$

which is a function of s. By differentiating P_d with respect to s and setting the derivative to zero, we can find the slip for the maximum power, which is given by

$$s_{\max,p} = \frac{R_2}{R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}$$

and

$$P_{d,\max} = \frac{3}{2}\frac{V_1^2}{R_1+R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}$$

The Efficiency

Using the approximate equivalent circuit, the output power can also be calculated by

$$P_o = P_d - P_r = P_{ag} - P_{rcu} - P_r = P_{in} - P_{scu} - P_{rcu} - P_r = 3V_1I_2\cos\theta - 3I_2^2R_1 - 3I_2^2R_2 - P_r =$$

Therefore, the efficiency is

$$\eta = \frac{P_o}{P_{in}} = \frac{3V_1I_2\cos\theta-3I_2^2(R_1+R_2)-P_r}{3V_1I_2\cos\theta}$$

Differentiating η with respect to I_2 and setting the derivative equal to zero gives

$$3I_2^2(R_1+R_2) = P_r$$

which implies that the efficiency is maximum when the sum of the stator and the rotor cooper losses is equal to the rotational loss, that is,

$$\eta_{\max} = \frac{3V_1I_2\cos\theta-2P_r}{3V_1I_2\cos\theta}$$

at

$$I_{2,\max,\eta} = \sqrt{\frac{P_r}{3I_2(R_1+R_2)}}$$

The Developed Torque

Note that $P_d = \omega_m\tau_d$. Thus the developed torque is given by

$$\tau_d = \frac{P_d}{\omega_m} = \frac{\frac{3V_1^2\frac{(1-s)R_2}{s}}{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}}{\omega_m} = \frac{3V_1^2\frac{(1-s)R_2}{s}}{(1-s)\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]} = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]}$$

Differentiating τ_d with respect to s and setting it equal to zero, it can be shown that the slip for the maximum torque is given by

$$s_{\max,\tau} = \frac{R_2}{\sqrt{R_1^2+(X_1+X_2)^2}}$$

and the corresponding maximum torque (pull-out torque or break-down torque) is given by

$$\tau_{d,\max} = \frac{3V_1^2}{2\omega_s\left[R_1+\sqrt{R_1^2+(X_1+X_2)^2}\right]}$$

Example 6.1: The following test data were obtained on a 460V, 4-pole, 60Hz, △-connected three-phase induction motor:

No-load test: power input=380W, line current = 1.15A at rated voltage.

Blocked-rotor test: power input=15W, line current = 2.1A at the line voltage of 21V.

The friction and windage loss is 21W, and the winding resistance between any two lines is 1.2Ω.

Determine (a) the equivalent circuit parameters of the motor, (b) the starting torque and starting current by using the approximate equivalent circuit, (c) the motor speed, developed torque, and efficiency at s=5%, (d) the maximum torque and its corresponding speed, (e) the maximum developed power and its corresponding speed, and (f) plot the developed torque against the slip.

Solution:

(a) The per–phase resistance of the stator is $R_1 = 1.5R_L = 1.5 \times 1.2 = 1.8\Omega$.

From the blocked-rotor test, $V_{br} = 21V$, $I_{br} = \frac{2.1}{\sqrt{3}} = 1.2A$, $P_{br} = \frac{15}{3} = 5W$. Therefore, the equivalent resistance is

$$R_e = \frac{P_{br}}{I_{br}^2} = \frac{5}{1.2^2} = 3.5\Omega$$

The rotor resistance is $R_2 = R_e - R_1 = 3.5 - 1.8 = 1.7\Omega$

The equivalent impedance is

$$|Z_e| = \frac{V_{br}}{I_{br}} = \frac{21}{1.2} = 17.5\Omega$$

The equivalent reactance is

$$X_e = \sqrt{|Z_e|^2 - R_e^2} = \sqrt{17.5^2 - 3.5^2} = 17.1\Omega$$

From the no-load test, $V_{NL} = 460V$, $I_{NL} = \frac{1.15}{\sqrt{3}} = 0.66A$, $P_{NL} = \frac{380}{3} = 127W$, $P_c = \frac{380-21}{3} = 120W$. Therefore, the equivalent core resistance is

$$R_c = \frac{V_{NL}^2}{P_c} = \frac{460^2}{120} = 1763.3\Omega$$

The excitation impedance is

$$|Z_\phi| = \frac{V_{NL}}{I_{NL}} = \frac{460}{0.66} = 696.97\Omega$$

The magnetization reactance is

$$X_m = \frac{1}{\sqrt{\left|\frac{1}{Z_\phi}\right|^2-\left(\frac{1}{R_c}\right)^2}} = \frac{1}{\sqrt{\left(\frac{1}{696.97}\right)^2-\left(\frac{1}{1763.3}\right)^2}} = 758.76\Omega$$

(b)
The synchronous speed is

$$\omega_s = \frac{4\pi f}{P} = \frac{4\pi\times 60}{4} = 188.5 rad/s \text{ or } \quad N_s = \frac{120f}{P} = \frac{120\times 60}{4} = 1800 rpm$$

The phase input voltage is $\widehat{V}_1 = 460\angle 0^\circ V$.
The starting torque is

$$\tau_d\big|_{s=1} = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]}\Bigg|_{s=1} = \frac{3\times 460^2\times 1.7}{188.5\left((3.5+0)^2+(17.1)^2\right)} = 18.8 N\bullet m$$

It follows from the approximate equivalent circuit that the rotor current at the time of starting is

$$\widehat{I}_2 = \frac{\widehat{V}_1}{R_1+R_2+j(X_1+X_2)+\frac{(1-s)R_2}{s}} = \frac{460\angle 0^\circ}{3.5+j17.1+0} = \frac{460(3.5-j17.1)}{(3.5+j17.1)(3.5-j17.1)} = \frac{460(3.5-j17.1)}{3.5^2+17.1^2} = 5.3-j25.8A$$

The stator current at the time of starting is

$$\widehat{I}_1 = \widehat{I}_\phi + \widehat{I}_2 = \frac{\widehat{V}_1}{R_c} + \frac{\widehat{V}_1}{jX_m} + \widehat{I}_2 = \widehat{V}_1\left(\frac{1}{R_c}+\frac{1}{jX_m}\right) + \widehat{I}_2$$

$$= (460\angle 0^\circ)\left(\frac{1}{1763.3}+\frac{1}{j758.76}\right) + (5.3-j25.8)$$

$$= (460)\left(\frac{1}{1763.3} - j\frac{1}{758.76}\right) + (5.3-j25.8)$$

$$= 5.56 - j26.41 = \sqrt{5.56^2+26.41^2}\angle\tan^{-1}\left(\frac{-26.41}{5.56}\right)\frac{180}{\pi} = 27.0\angle -27.0^\circ A$$

(c)
The motor speed at s=0.05 is $\omega_m = (1-s)\omega_s = (1-0.05)\times 188.5 = 179.1 rad/s$ or $N_m = (1-s)N_s = (1-0.05)\times 1800 = 1710 rpm$.
The developed torque is

$$\tau_d\big|_{s=5\%} = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]}\Bigg|_{s=5\%} = \frac{3\times 460^2\times 1.7}{0.05\times 188.5\left(\left(3.5+\frac{(1-0.05)\times 1.7}{0.05}\right)^2+(17.1)^2\right)} = 72.7 N\bullet m$$

It follows from the approximate equivalent circuit that the rotor current at s=5% is

$$\widehat{I}_2 = \frac{\widehat{V}_1}{R_1+R_2+j(X_1+X_2)+\frac{(1-s)R_2}{s}} = \frac{460\angle 0^\circ}{3.5+j17.1+\frac{(1-0.05)\times 1.7}{0.05}} = \frac{460(35.8-j17.1)}{(35.8+j17.1)(35.8-j17.1)} = \frac{460(35.8-j17.1)}{35.8^2+17.1^2}$$

$$= 10.46 - j5.0 = \sqrt{10.46^2+5.0^2}\angle\tan^{-1}\left(\frac{-5.0}{10.46}\right)\frac{180}{\pi} = 11.6\angle -25.5^\circ A$$

The stator current at the time of starting is

$$\widehat{I}_1 = \widehat{I}_\phi + \widehat{I}_2 = \frac{\widehat{V}_1}{R_c} + \frac{\widehat{V}_1}{jX_m} + \widehat{I}_2 = \widehat{V}_1\left(\frac{1}{R_c}+\frac{1}{jX_m}\right) + \widehat{I}_2$$

$$= (460\angle 0^\circ)\left(\frac{1}{1763.3}+\frac{1}{j758.76}\right) + (10.46-j5.0)$$

$$= (460)\left(\frac{1}{1763.3} - j\frac{1}{758.76}\right) + (10.46-j5.0)$$

$$= 10.72 - j5.61 = \sqrt{10.72^2+5.61^2}\angle\tan^{-1}\left(\frac{-5.61}{10.72}\right)\frac{180}{\pi} = 12.1\angle -27.6^\circ A$$

The input power is

$$P_{in} = 3V_1I_1\cos\theta = 3\times 460\times 12.1\times\cos\left(27.6\frac{\pi}{180}\right) = 14798W$$

The stator copper loss is $P_{scu} = 3I_2^2R_1 = 3\times 11.6^2\times 1.8 = 727W$
The air-gap power is $P_{ag} = P_{in} - P_{scu} = 14798 - 727 = 14071W$
The rotor copper loss: $P_{rcu} = 3I_2^2R_2 = 3\times 11.6^2\times 1.7 = 686W$

The developed power: $P_d = P_{ag} - P_{rcu} = 14071 - 686 = 13385W$
The output power: $P_o = P_d - P_r = P_d - P_c - P_{fw} = 13385 - (380 - 21) - 21 = 13005W$
The efficiency is

$$\eta = \frac{P_o}{P_{in}} \times 100 = \frac{13005}{14798} \times 100 = 87.9\%$$

(d)
The slip for the maximum torque is

$$s_{\max,\tau} = \frac{R_2}{\sqrt{R_1^2+(X_1+X_2)^2}} = \frac{1.7}{\sqrt{1.8^2+(17.1)^2}} = 0.099$$

and the corresponding maximum torque (pull-out torque or break-down torque) is given by

$$\tau_{d,\max} = \frac{3V_1^2}{2\omega_s\left[R_1+\sqrt{R_1^2+(X_1+X_2)^2}\right]} = \frac{3\times460^2}{2\times188.5\times\left(1.8+\sqrt{1.8^2+(17.1)^2}\right)} = 88.6N \cdot m$$

The speed is

$$N_m = (1-s)N_s = (1-0.099)\times 1800 = 1622rpm$$

(e)
The slip for the maximum developed power is

$$s_{\max,p} = \frac{R_2}{R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}} = \frac{1.7}{1.7+\sqrt{3.5^2+17.1^2}} = 0.089$$

and

$$P_{d,\max} = \frac{3}{2}\frac{V_1^2}{R_1+R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}} = \frac{3}{2}\frac{460^2}{3.5+\sqrt{3.5^2+17.1^2}} = 15147W$$

The speed is

$$N_m = (1-s)N_s = (1-0.089)\times 1800 = 1640rpm$$

or

$$\omega_m = (1-s)\omega_s = (1-0.089)\times 188.5 = 171.7rad/s$$

The developed torque is

$$\tau_d = \frac{P_{d,\max}}{\omega_m} = \frac{15147}{171.7} = 88.2N \cdot m$$

(f) The relationship between the developed torque and the slip is given by

$$\tau_d = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2\right]} = \frac{3\times460^2\times1.7}{s\times188.5\times\left(\left(3.5+\frac{(1-s)\times1.7}{s}\right)^2+(17.1)^2\right)} =$$

$$\frac{5725.0}{s\left((\frac{1}{s}(1.7s-1.7)-3.5)^2+292.41\right)}$$

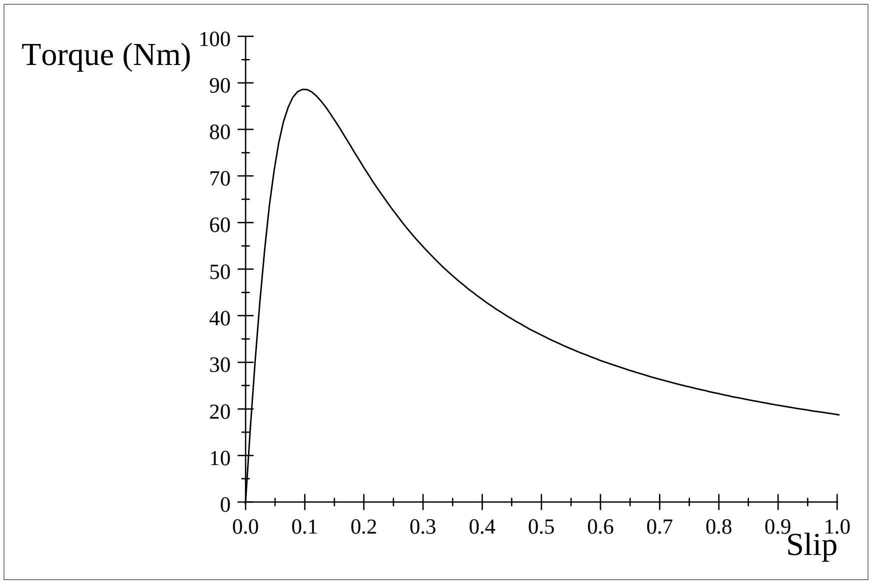


Figure 6.6

Some Important Observation:

(1) When the motor is operating near its rated slip, which is less than 10%, the developed torque is directly proportional slip.

(2) For a constant slip, the developed torque is directly proportional the square of the applied voltage.

6.2 Single-Phase Induction Motors

6.2.1 Double Revolving-Field Theory

Similar to three-phase induction motors, single-phase motors have the stator and rotor with the rotor winding short-circuited. Unlike three-phase induction motors, single-phase induction motors have only one phase winding in the stator. A single-phase AC voltage is applied to the stator winding, which produces an AC current in the stator winding. Suppose that $i(t) = I_m \cos(\omega t)$. Then, the resultant air-gap magnetic flux density is given by

$$\vec{B} = B_{\max} \cos(\omega t) \vec{i} = \vec{B}_{CW} + \vec{B}_{CCW}$$

where $\vec{B}_{CW}$ and $\vec{B}_{CCW}$ represent the clockwise and counterclockwise rotating magnetic fields, respectively, defined by

$$\vec{B}_{CW} = 0.5B_m \cos(\omega t) \vec{i} - 0.5B_m \sin(\omega t) \vec{j}$$

$$\vec{B}_{CCW} = 0.5B_m \cos(\omega t) \vec{i} + 0.5B_m \sin(\omega t) \vec{j}$$

The equation above implies that the sum of the clockwise and counterclockwise rotating magnetic fields is equal to the stationary pulsating megnetic field, that is, the stationary pulsating megnetic field can be resolved into two rotating magnetic fields, each of equal magnetude but rotating at the synchronous speed in opposite directions. The synchronous speed is determined by

$\omega_s = \frac{4\pi f}{P}$ or $N_s = \frac{120f}{P}$

Suppose that the motor rotates in the counterclockwise direction at speed ω_m. Define $s = \frac{\omega_s - \omega_m}{\omega_s}$. Then, $\vec{B}_{CCW}$ is called the forward magnetic field, denoted $\vec{B}_f$, rotating at the synchronous speed of $\omega_{fs} = \omega_s$ while $\vec{B}_{CW}$ the backward magnetic field at $\omega_{bs} = -\omega_s$, denoted $\vec{B}_b$. The slip of the motor is $s_f = \frac{\omega_{fs} - \omega_m}{\omega_{fs}} = s$ with respect to $\vec{B}_f$ and $s_b = \frac{\omega_{rs} - \omega_m}{\omega_{rs}} = \frac{-\omega_s - \omega_m}{-\omega_s} = \frac{-2\omega_s + \omega_s - \omega_m}{-\omega_s} = 2 - s$ with respect to $\vec{B}_b$.

Similar to three-phase induction motors, single-phase motors can be analyzed by using the equivalent circuit. Figure 6.7 shows the equivalent circuit for a single-phase AC motor at still, which is equivalent to the circuit with the effects of the forward and backward magnetic fields separated, as shown in Figure 6.8. For a motor running at speed ω_m, the effective rotor resistance changes with the slip. The rotor resistance is $\frac{R_2}{s}$ with respect to $\vec{B}_f$ while $\frac{R_2}{2-s}$ with respect to $\vec{B}_b$. The final equivalent circuit is shown in Figure 6.9.

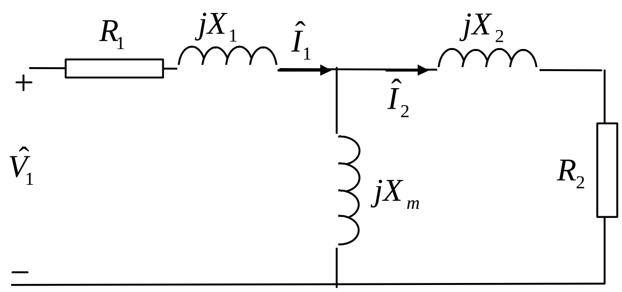


Figure 6.7

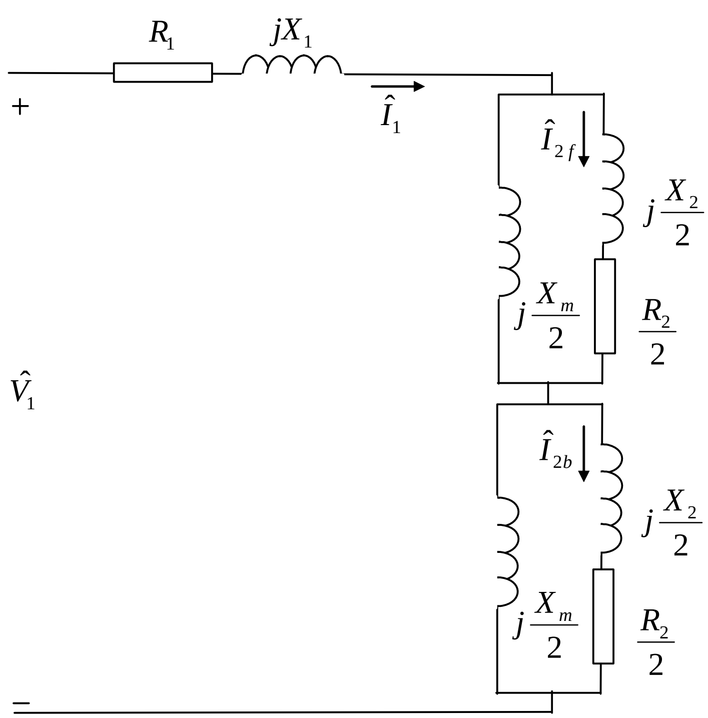


Figure 6.8

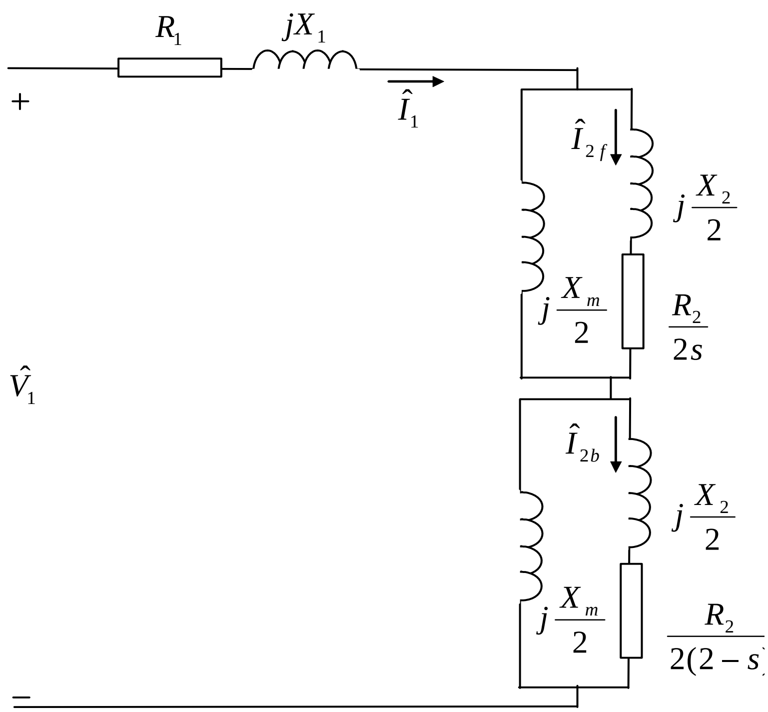


Figure 6.9

Define

$$Z_1 = R_1 + jX_1$$

$$Z_f = R_f + jX_f = 0.5\frac{jX_m(R_2/s+jX_2)}{R_2/s+j(X_2+X_m)} = 0.5\frac{jX_m(R_2/s+jX_2)((R_2/s-j(X_2+X_m)))}{(R_2/s+j(X_2+X_m))(R_2/s-j(X_2+X_m))}$$

$$= 0.5\frac{jX_m\left((R_2/s)^2+X_2(X_2+X_m)-jX_m(R_2/s)\right)}{(R_2/s)^2+(X_2+X_m)^2} = 0.5\frac{X_m^2(R_2/s)+jX_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$$

$$= 0.5\frac{X_m^2(R_2/s)}{(R_2/s)^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$$

$$Z_b = R_b + jX_b = 0.5\frac{jX_m(R_2/(2-s)+jX_2)}{R_2/(2-s)+j(X_2+X_m)} = 0.5\frac{jX_m(R_2/(2-s)+jX_2)((R_2/(2-s)-j(X_2+X_m)))}{(R_2/(2-s)+j(X_2+X_m))(R_2/(2-s)-j(X_2+X_m))}$$

$$= 0.5\frac{jX_m\left((R_2/(2-s))^2+X_2(X_2+X_m)-jX_m(R_2/(2-s))\right)}{(R_2/(2-s))^2+(X_2+X_m)^2} = 0.5\frac{X_m^2(R_2/(2-s))+jX_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$$

$$= 0.5\frac{X_m^2(R_2/(2-s))}{(R_2/(2-s))^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$$

$$R_f = 0.5\frac{X_m^2(R_2/s)}{(R_2/s)^2+(X_2+X_m)^2}$$

$$X_f = 0.5\frac{X_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$$

$$R_b = 0.5\frac{X_m^2(R_2/(2-s))}{(R_2/(2-s))^2+(X_2+X_m)^2}$$

$$X_b = 0.5\frac{X_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$$

Then,

$$Z_{in} = Z_1 + Z_f + Z_b$$

Thus, the stator current is

$$\hat{I}_1 = \frac{\hat{V}_1}{Z_{in}}$$

The input power is

$$P_{in} = \mathrm{Re}\left[\hat{V}_1\hat{I}_1^*\right] = V_1I_1\cos\theta$$

The stator copper loss is

$$P_{scu} = I_1^2R_1$$

The air-gap power due to the forward magnetic field is

$$P_{agf} = I_1^2R_f = 0.5I_{2f}^2\frac{R_2}{s}$$

The air-gap power due to the backward magnetic field is

$$P_{agb} = I_1^2R_b = 0.5I_{2b}^2\frac{R_2}{2-s}$$

The forward rotor copper loss is

$$P_{rcuf} = 0.5I_{2f}^2R_2 = sP_{agf}$$

The backward rotor copper loss is

$$P_{rcub} = 0.5I_{2b}^2R_2 = (2-s)P_{agb}$$

The power developed by the forward magnetic field is

$$P_{df} = P_{agf} - P_{rcuf} = (1-s)P_{agf}$$

The power developed by the backward magnetic field is

$$P_{db} = P_{agb} - P_{rcub} = -(1-s)P_{agb}$$

The total developed power is

$$P_d = P_{df} + P_{db} = (1-s)(P_{agf} - P_{agb}) = (1-s)P_{ag}$$

So, the net air-gap power is

$$P_{ag} = P_{agf} - P_{agb}$$

The mechanical developed power is

$$P_d = (1-s)P_{ag} = \tau_d\omega_m = (1-s)\tau_d\omega_s$$

The developed torque is

$$\tau_d = \frac{P_d}{\omega_m} = \frac{(1-s)P_{ag}}{(1-s)\omega_s} = \frac{P_{agf}-P_{agb}}{\omega_s} = \frac{P_{agf}}{\omega_s} - \frac{P_{agb}}{\omega_s} = \tau_{fd} - \tau_{bd}$$

The output power is

$$P_o = P_d - P_r$$

where the rotational loss is $P_r = P_c + P_{fw} + P_{stray}$.

Example 6.2: A 4-pole 110V 50Hz single-phase induction motor has $R_1 = 2\Omega$, $X_1 = 2.8\Omega$, $R_2 = 3.8\Omega$, $X_2 = 2.8\Omega$, and $X_m = 60\Omega$. The rotational loss is 20W. Determine the shaft torque, the motor efficiency when the slip is 4%, and the developed torque characteristics.

Solution:The synchronous speed is

$$\omega_s = \frac{4\pi f}{P} = \frac{4\pi 50}{4} = 157.08rad/s \text{ or } N_s = \frac{120f}{P} = \frac{120\times 50}{4} = 1500rpm$$

The impedances are

$$Z_1 = R_1 + jX_1 = 2 + j2.8$$

$$Z_f = R_f + jX_f = \frac{0.5jX_m(0.5R_2/s + j0.5X_2)}{0.5jX_m + (0.5R_2/s + j0.5X_2)} = 0.5\frac{jX_m(R_2/s + jX_2)}{R_2/s + j(X_2 + X_m)} = 0.5\frac{j60(3.8/0.04 + j2.8)}{3.8/0.04 + j(2.8+60)}$$

$$= 0.5\frac{-168 + j5700}{95.0 + j62.8} = 0.5\frac{(-168 + j5700)(95 - j62.8)}{(95 + j62.8)(95 - j62.8)} = 0.5\frac{3.42\times 10^5 + j5.5205\times 10^5}{95^2 + 62.8^2} = 13.185 + j21.284\Omega$$

$$Z_b = R_b + jX_b = \frac{0.5jX_m[0.5R_2/(2-s) + j0.5X_2]}{0.5jX_m + [0.5R_2/(2-s) + j0.5X_2]} = 0.5\frac{jX_m[R_2/(2-s) + jX_2]}{R_2/(2-s) + j(X_2 + X_m)} = 0.5\frac{j60\times(3.8/(2-0.04) + j2.8)}{3.8/(2-0.04) + j(2.8+60)}$$

$= 0.5\frac{-168.0+j116.33}{1.9388+j62.8} = 0.5\frac{(-168.0+j116.33)(1.9388-j62.8)}{(1.9388+j62.8)(1.9388-j62.8)} = \frac{0.5(6979.8+j10776)}{1.9388^2+62.8^2} = 0.88406+j1.3649\,\Omega$

$Z_{in} = Z_1 + Z_f + Z_b = (2+j2.8)+(13.185+j21.284)+(0.88406+j1.3649) = 16.069+j25.449\Omega$

Thus, the stator current is

$$\hat{I}_1 = \frac{\hat{V}_1}{Z_{in}} = \frac{110\angle 0^\circ}{16.069+j25.449} = \frac{110\angle 0^\circ}{\sqrt{16.069^2+25.449^2}\angle \tan^{-1}\left(\frac{25.449}{16.069}\right)\frac{180}{\pi}} = \frac{110\angle 0^\circ}{30.098\angle 57.731^\circ} = 3.6547\angle -57.731^\circ A$$

$$\hat{I}_{2f} = \frac{j\frac{X_m}{2}}{\frac{R_2}{2s}+j\left(\frac{X_2}{2}+\frac{X_m}{2}\right)}\hat{I}_1 = \frac{jX_m}{\frac{R_2}{s}+j(X_2+X_m)}\hat{I}_1 = \frac{j60}{\frac{3.8}{0.04}+j(2.8+60)}(3.6547\angle -57.731^\circ)$$

$$= \frac{(60\angle 90^\circ)(3.6547\angle -57.731^\circ)}{\sqrt{\left(\frac{3.8}{0.04}\right)^2+(2.8+60)^2}\angle \tan^{-1}\left(\frac{2.8+60}{\frac{3.8}{0.04}}\right)\frac{180}{\pi}} = \frac{(60\angle 90^\circ)(3.6547\angle -57.731^\circ)}{113.88\angle 33.467^\circ} = 1.9256\angle -1.198^\circ A$$

$$\hat{I}_{2b} = \frac{j\frac{X_m}{2}}{\frac{R_2}{2(2-s)}+j\left(\frac{X_2}{2}+\frac{X_m}{2}\right)}\hat{I}_1 = \frac{jX_m}{\frac{R_2}{s}+j(X_2+X_m)}\hat{I}_1 = \frac{j60}{\frac{3.8}{2-0.04}+j(2.8+60)}(3.6547\angle -57.731^\circ)$$

$$= \frac{(60\angle 90^\circ)(3.6547\angle -57.731^\circ)}{\sqrt{\left(\frac{3.8}{2-0.04}\right)^2+(2.8+60)^2}\angle \tan^{-1}\left(\frac{2.8+60}{\frac{3.8}{2-0.04}}\right)\frac{180}{\pi}} = \frac{(60\angle 90^\circ)(3.6547\angle -57.731^\circ)}{62.830\angle 88.232^\circ} = 3.4901\angle -55.963^\circ A$$

The input power is

$$P_{in} = \mathrm{Re}\left[\hat{V}_1\hat{I}_1^*\right] = V_1 I_1\cos\theta = 110\times 3.6547\times\cos\left(57.731\frac{\pi}{180}\right) = 214.63W$$

The stator copper loss is

$$P_{scu} = I_1^2 R_1 = 3.6547^2\times 2 = 26.714W$$

The air-gap power due to the forward magnetic field is

$P_{agf} = I_1^2 R_f = 3.6547^2\times 13.185 = 176.11W$ or $= 0.5I_{2f}^2\frac{R_2}{s} = 0.5\times 1.9256^2\frac{3.8}{0.04} = 176.13W$

The air-gap power due to the backward magnetic field is

$P_{agb} = I_1^2 R_b = 3.6547^2\times 0.88406 = 11.808W$ or $= 0.5I_{2b}^2\frac{R_2}{2-s} = 0.5\times 3.4901^2\times\frac{3.8}{2-0.04} = 11.808W$

The net air-gap power is

$$P_{ag} = P_{agf} - P_{agb} = 176.13 - 11.808 = 164.32W$$

The mechanical developed power is

$$P_d = (1-s)P_{ag} = (1-0.04)\times 164.32 = 157.75W$$

The output power is

$$P_o = P_d - P_r = 157.75 - 20 = 137.75W$$

The efficiency is

$$\eta = \frac{137.75}{214.63}\times 100 = 64.2\%$$

The motor speed is

$$\omega_m = (1-s)\omega_s = (1-0.04)\times 157.08 = 150.80rad/s$$

The motor shaft torque is

$$\tau_o = \frac{P_o}{\omega_m} = \frac{137.75}{150.80} = 0.91346N\cdot m$$

The developed torque of the forward and backward magnetic field is

$$\tau_{df} = \frac{(1-s)P_{agf}}{\omega_m} = \frac{V_1^2 R_f}{\omega_s\left((R_1+R_f+R_b)^2+(X_1+X_f+X_b)^2\right)} = \frac{110^2 R_f}{157.08\left((2+R_f+R_b)^2+(2.8+X_f+X_b)^2\right)}$$

$$\tau_{db} = \frac{V_1^2 R_b}{\omega_s\left((R_1+R_f+R_b)^2+(X_1+X_f+X_b)^2\right)} = \frac{110^2 R_b}{157.08\left((2+R_f+R_b)^2+(2.8+X_f+X_b)^2\right)}$$

with

$$R_f = 0.5\frac{60^2(3.8/s)}{(3.8/s)^2+(2.8+60)^2}$$

$$X_f = 0.5\frac{60\left((3.8/s)^2+2.8(2.8+60)\right)}{(3.8/s)^2+(2.8+60)^2}$$

$$R_b = 0.5\frac{60^2(3.8/(2-s))}{(3.8/(2-s))^2+(2.8+60)^2}$$

$$X_b = 0.5\frac{60\left((3.8/(2-s))^2+2.8(2.8+60)\right)}{(3.8/(2-s))^2+(2.8+60)^2}$$

$$\tau_d = \frac{110^2 R_f}{157.08\left((2+R_f+R_b)^2+(2.8+X_f+X_b)^2\right)} - \frac{110^2 R_b}{157.08\left((2+R_f+R_b)^2+(2.8+X_f+X_b)^2\right)}$$

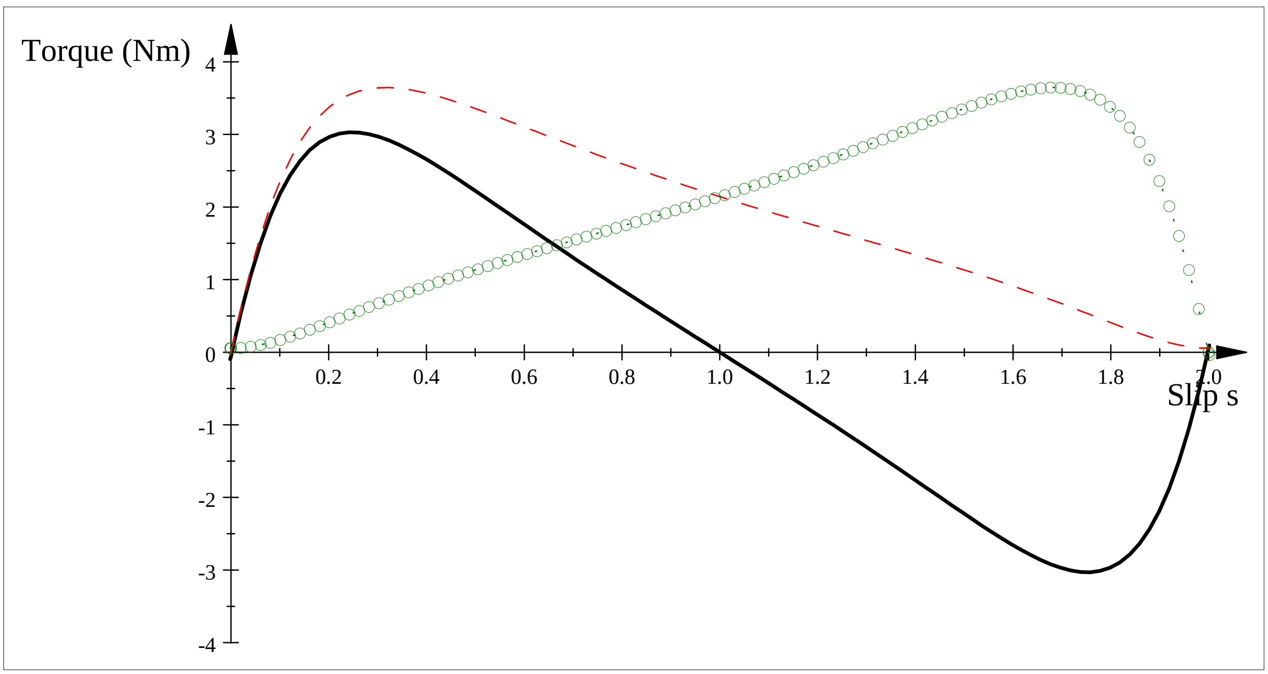


Figure 6.10 τ_d (thick solid line) τ_{df} (dashed line); τ_{df} (dot dot dashed line)

6.2.2 Types of Single-Phase Induction Motors

It is observed from Figure 6.10 that the developed torque is the torque developed by the forward magnetic field less the torque developed by the backward magnetic field. It is noted that τ_{df} and τ_{db} are the same at the starting moment, so the starting torque is zero, which means that this motor cannot start by itself. However, by introducing an extra winding and some capacitors, single-phase induction motors can be made self-starting.

1. Split-Phase Motors

A split-phase induction motor has two separate windings: main winding and auxiliary winding. They are placed in space quadrature and connected to a single-phase power source. The main winding has a low resistance and high inductance and carries current to establish the main flux at the rated speed. The auxiliary winding has a high resistance and low inductance and is desconnected from the supply by a centrifugal switch when the motor reaches a speed of nearly 75% of its synchronous speed.

At the time of starting, the main winding current lags the applied voltage by almost 90° owing to its high inductance (large number of turns) and low resistance (large size wire) while the auxiliary winding current is essentially in phase with the applied voltage due to its low inductance and high resistance. Since the two windings are placed in space quadrature and carry out-of-phase currents, a rotating magnetic field is produced in the air-gap and the motor is able to rotate by itself.

2. Capacitor-Start Motors

In split-phase motor, the main winding current does not lag the auxiliary winding current exactly by 90°. However, by connecting a capacitor in series with the auxiliary winding, it is possible to make the main winding current lag the auxiliary winding current exactly by 90°.

3. Capacitor-Start Capacitor Run Motors

The power factor for both split-phase and capacitor start motors is low and so is efficiency, usually 50%-60%. The efficiency can be improved by employing another capacitor when the motor runs at the rated speed. This led to the development of a capacitor-start and capacitor-run motor.

4. Permanent Split-Capacitor Motors

The permanent split-capacitor motor is developed by removing the start-capacitor and cetrifugal switch from the capacitor-start capacitor-run motor.

Chapter 7. Special Motors

7.1 Universal Motors

A DC series motor specially designed for AC operation is usually referred to as a universal motor. The equivalent circuit is shown in Figure 7.1

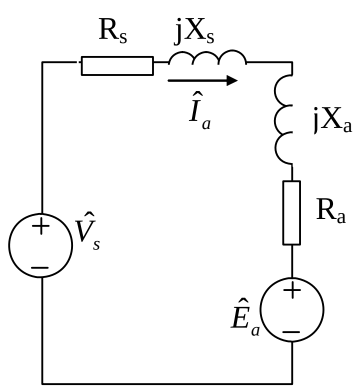


Figure 7.1

The phasor diagram for a lagging load is shown in Figure 7.2.

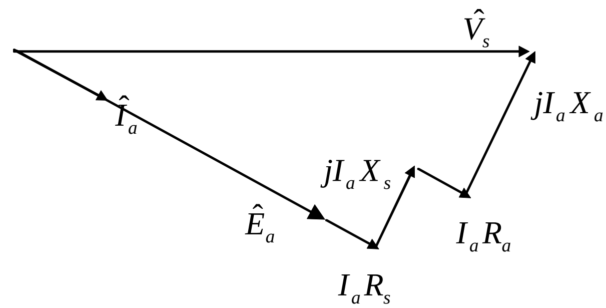


Figure 7.2

Example 7.1: A 120V 60Hz 2-pole universal motor operates at a speed of 8000rpm on full load and draws a current of 17.58A at a lagging power factor of 0.912. The impedance of the series field winding is 0.65+j1.2Ω. The impedance of the armature winding is 1.36+j1.6Ω. Determine (a) the induced voltage, (b) the power output, (c) the shaft torque, and (d) the efficiency if the rotational loss is 80W.

Solution: From the equivalent circuit, we have

$$\widehat{E}_a = \widehat{V}_s - \widehat{I}_a(R_s + R_a + jX_s + jX_a) = 120 - (17.58\angle -24.22^\circ)(0.65 + 1.36 + j(1.2 + 1.6))$$
$$= 120 - 17.58\left(\cos\left(-24.22\tfrac{\pi}{180}\right) + j\sin\left(-24.22\tfrac{\pi}{180}\right)\right)(0.65 + 1.36 + j(1.2 + 1.6))$$
$$= 67.581 - j30.395 = \sqrt{67.581^2 + 30.395^2}\angle \tan^{-1}\left(\tfrac{-30.395}{67.581}\right)\tfrac{180}{\pi} = 74.1\angle -24.22^\circ V$$

Note that the induced voltage is in phase with the armature current.

The input power is

$$P_{in} = V_s I_a \cos\theta = 120 \times 17.58 \times 0.912 = 1924W$$

The copper loss

$$P_{cu} = I_a^2(R_s + R_a) = 17.58^2(0.65 + 1.36) = 621.2W$$

The developed power is

$$P_d = P_{in} - P_{cu} = 1924 - 621.2 = 1302.8W$$

The output power is

$$P_o = P_d - P_r = 1302.8 - 80 = 1222.8W$$

The efficiency is

$$\eta = \tfrac{P_o}{P_{in}} \times 100 = \tfrac{1222.8}{1924} \times 100 = 63.6\%$$

The motor speed is

$$\omega_m = \tfrac{2\pi n}{60} = \tfrac{2\pi \times 8000}{60} = 837.76 rad/s$$

The shaft torque is

$$\tau_o = \tfrac{P_o}{\omega_m} = \tfrac{1222.8}{837.76} = 1.46N \cdot m$$

7.2 Permanent DC Motors

A DC motor with the magnetic field being produced by permanent magnets is called the permanent DC motor. The equivalent circuit for a permanent DC motor is shown in Figure 7.3.

The dynamical equations are given by

$$e_a(t) = K_a\Phi_a\omega(t)$$
$$\tau_d(t) = K_a\Phi_a i_a(t)$$
$$v_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + K_a\Phi_a\omega(t)$$
$$J\frac{d\omega(t)}{dt} = K_a\Phi_a i_a(t) - \tau_L(t) - D\omega(t)$$

The steady-state quantities are calculated by letting $\frac{di_a(t)}{dt}$ and $\frac{d\omega(t)}{dt}$ be zero, that is,

$$e_a(\infty) = K_a\Phi_a\omega(\infty)$$
$$\tau_d(\infty) = K_a\Phi_a i_a(\infty)$$
$$v_a(\infty) = R_a i_a(\infty) + K_a\Phi_a\omega(\infty)$$
$$0 = K_a\Phi_a i_a(\infty) - \tau_L(\infty) - D\omega(\infty)$$

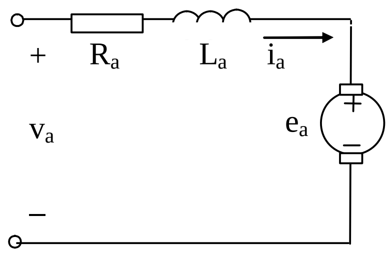


Figure 7.3

Example 7.2: Calculate the magnetic flux in a 200W, 100V PM DC motor operating at 1500rpm. The motor constant is 85, the armature resistnace is 2Ω, and the rotational loss is 15W.

Solution: $\omega(\infty) = \frac{2\pi n}{60} = \frac{2\pi\times1500}{60} = 157.08 rad/s$

The developed power is $P_d = P_o + P_r = 200 + 15 = 215W$

The developed torque is $\tau_d(\infty) = \frac{P_d}{\omega(\infty)} = \frac{215}{157.08} = 1.3687 N \cdot m$

It follows from $\tau_d(\infty) = K_a\Phi_a i_a(\infty)$ that

$$i_a(\infty) = \frac{\tau_d(\infty)}{K_a\Phi_a}$$

Substituting this into $v_a(\infty) = R_a i_a(\infty) + K_a\Phi_a\omega(\infty)$ gives

$$v_a(\infty) = R_a\frac{\tau_d(\infty)}{K_a\Phi_a} + K_a\Phi_a\omega(\infty)$$

that is,

$$100 = 2\frac{1.3687}{85\Phi_a} + 85 \times 157.08\Phi_a^2$$

or

$$8500\Phi_a = 2 \times 1.3687 + 85^2 \times 157.08\Phi_a^2$$

Solving this for positive Φ_a produces

$$\Phi_a = \frac{8500 \pm \sqrt{8500^2 - 4\times2\times1.3687\times85^2\times157.08}}{2\times85^2\times157.08} = 7.1524\times10^{-3} \text{ and } 3.3723\times10^{-4}$$

Because $e_a(\infty) = K_a\Phi_a\omega(\infty) = 85 \times 3.3723 \times 10^{-4} \times 157.08 = 4.5026V$ is too small and $e_a(\infty) = K_a\Phi_a\omega(\infty) = 85 \times 7.1524 \times 10^{-3} \times 157.08 = 95.497V$ is reasonable, so $\Phi_a = 7.1524 \times 10^{-3} Wb$.

7.3 Stepper Motors

$$\theta_m = \frac{2}{P}\theta_e$$
$$\omega_m = \frac{2}{P}\omega_e$$

$$n_m = \frac{2}{P} n_e$$
$$n_e = \frac{1}{2N} n_{pulses}$$
$$n_m = \frac{1}{NP} n_{pulses}$$

where P is the number of poles, N is the number of phases, θ_m is the mechanical angle, θ_e is the electrical angle, ω_m and n_m are the mechanical speed, ω_e and n_e are the electrical speed, n_{pulses} is the number of pules per minute.

Example 7.3: A three-phase permanent-magnet stepper motor required for one particular application must be capable of controlling the position of a shaft in steps of 7.5°, and it nust be capable of running at speeds of up to 300rpm. (a) How many poles must this motor have? (b) At what rate must control pulses be received in the motor's control unit if it is to be driven at 300rpm?

Solution: (a) In a three-phase stepper motor, each pulse advances the rotor's position by 60 electrical degrees. This advance must correspond to 7.5 mechanical degrees. Solving $\theta_m = \frac{2}{P}\theta_e$ for P yields

$$P = 2\frac{\theta_e}{\theta_m} = 2\frac{60}{7.5} = 16 \text{ poles}$$

(b) Solving $n_m = \frac{1}{NP} n_{pulses}$ for n_{pulses} gives

$$n_{pulses} = NPn_m = 3 \times 16 \times 300 = 14400 \text{ pulses/minute=240 pulses/s}$$

Formula Sheet for the Final Exam:

$\vec{B} = \mu\vec{H}, \phi = BA, \mu_0 = 4\pi \times 10^{-7} H/m$

$\phi = \frac{\mathcal{F}}{\Re}, \mathcal{F} = Ni, \Re = \frac{l}{\mu A}$

$e = \frac{d\lambda}{dt}, \lambda = N\phi, L = \frac{\lambda}{i} = \frac{N^2}{\Re}$

$W_\phi(\lambda, x) = \frac{1}{2L}\lambda^2, W_\phi(i, x) = \frac{1}{2}Li^2$

$f = -\frac{\partial W_\phi(\lambda,x)}{\partial x} = \frac{1}{2}\lambda^2 \frac{1}{L^2}\frac{dL(x)}{dx}, f = \frac{\partial W_\phi(i,x)}{\partial x} = \frac{1}{2}i^2\frac{dL(x)}{dx}$

$e_1 = \frac{d\lambda_1}{dt}, e_2 = \frac{d\lambda_2}{dt}, \lambda_1 = \lambda_{11} + \lambda_{12} = L_{11}i_1 + L_{12}i_2, \lambda_2 = \lambda_{21} + \lambda_{22} = L_{21}i_1 + L_{22}i_2$

$W_\phi(\lambda_1, \lambda_2, \theta) = \frac{1}{2}\Gamma_{11}^2\lambda_1^2 + \Gamma_{12}\lambda_1\lambda_2 + \frac{1}{2}\Gamma_{22}\lambda_2^2, W_\phi(i_1, i_2, \theta) = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1i_2 + \frac{1}{2}L_{22}i_2^2$

$\tau = -\frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial\theta} = \frac{1}{2}\lambda_1^2\frac{d\Gamma_{11}(\theta)}{d\theta} + \lambda_1\lambda_2\frac{d\Gamma_{12}(\theta)}{d\theta} + \frac{1}{2}\lambda_2^2\frac{d\Gamma_{22}(\theta)}{d\theta}$

$\tau = \frac{\partial W_\phi(i_1,i_2,\theta)}{\partial\theta} = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$

$e = l\vec{v} \times \vec{B}, f = i\vec{l} \times \vec{B}$

$e_a(t) = K_e i_f(t)\omega(t), \tau_d(t) = K_\tau i_f(t) i_a(t)$

$\omega(s) = \frac{K_e I_f(V_a(s)+L_a i_a(0)) + (L_a s+R_a)(J\omega(0)-\tau_L(s))}{(Js+D)(L_a s+R_a)+(K_e I_f)^2}$

$I_a(s) = \frac{(V_a(s)+L_a i_a(0))(Js+D) - K_e I_f(J\omega(0)-\tau_L(s))}{(Js+D)(L_a s+R_a)+(K_e I_f)^2}$

$I_f(s) = \frac{V_f(s)+L_f i_f(0)}{L_f s+R_f}$

$Z_Y = \frac{1}{3}Z_\triangle, \hat{V}_{an} = \frac{\hat{V}_{ab}}{\sqrt{3}}\angle -30^\circ, \hat{I}_{A'} = \sqrt{3}\,\hat{I}_A\angle -30^\circ$

$\triangle$-**Y connection**: $\hat{E}_{A_1} = a\hat{E}_{A_2}\angle -30^\circ$, $\hat{I}'_{A_2} = \frac{1}{a}\hat{I}_{A_2}\angle -30^\circ$

Y-$\triangle$ connection: $\hat{E}_{A_1} = a\hat{E}_{A_2}\angle 30^\circ$, $\hat{I}'_{A_2} = \frac{1}{a}\hat{I}_{A_2}\angle 30^\circ$

$\omega_m = (1-s)\omega_s, \omega_s = \frac{4\pi f}{P}$

$\hat{E}'_a = \hat{E}_a - j\hat{I}_d(X_d - X_q)$(syn. generator),$\hat{E}'_a = \hat{E}_a + j\hat{I}_d(X_d - X_q)$(syn. motor)

$P_d = 3I_2^2\frac{(1-s)R_2}{s} = \frac{3V_1^2\frac{(1-s)R_2}{s}}{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}, \tau_d = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+\frac{R_2}{s}\right)^2+(X_1+X_2)^2\right]}$

$s_{\max,p} = \frac{R_2}{R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}, P_{d,\max} = \frac{3}{2}\frac{V_1^2}{R_1+R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}$

$s_{\max,\tau} = \frac{R_2}{\sqrt{R_1^2+(X_1+X_2)^2}}, \tau_{d,\max} = \frac{3V_1^2}{2\omega_s\left[R_1+\sqrt{R_1^2+(X_1+X_2)^2}\right]}$

$Z_f = R_f + jX_f = 0.5\frac{jX_m(R_2/s+jX_2)}{R_2/s+j(X_2+X_m)} = 0.5\frac{X_m^2(R_2/s)}{(R_2/s)^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$

$Z_b = R_b + jX_b = 0.5\frac{jX_m(R_2/(2-s)+jX_2)}{R_2/(2-s)+j(X_2+X_m)} = 0.5\frac{X_m^2(R_2/(2-s))}{(R_2/(2-s))^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$

$P_{agf} = I_1^2R_f = 0.5I_{2f}^2\frac{R_2}{s}, P_{agb} = I_1^2R_b = 0.5I_{2b}^2\frac{R_2}{2-s}$

$P_{df} = P_{agf} - P_{rcuf} = (1-s)P_{agf}, P_{db} = P_{agb} - P_{rcub} = -(1-s)P_{agb}$

$P_d = (1-s)P_{ag}, P_{ag} = P_{agf} - P_{agb}$

$P_d = (1-s)P_{ag} = \tau_d\omega_m = (1-s)\tau_d\omega_s$

$\tau_d = \frac{P_{agf}}{\omega_s} - \frac{P_{agb}}{\omega_s} = \tau_{fd} - \tau_{bd}$

$e_a(t) = K_a\Phi_a\omega(t), \tau_d(t) = K_a\Phi_a i_a(t)$

$\theta_m = \frac{2}{P}\theta_e, n_m = \frac{1}{NP} n_{pulses}$

Review

Chapter 1. Magnetic Circuit

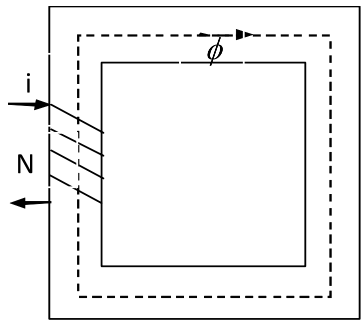


Figure R.1

$\vec{B} = \mu\vec{H}, \phi = BA, \mu_0 = 4\pi \times 10^{-7} H/m,$

$\phi = \frac{\mathcal{F}}{\Re}, \mathcal{F} = Ni, \Re = \frac{l}{\mu A}$

$\Re_{eq} = \Re_1 + \Re_2 + \cdots, \frac{1}{\Re_{eq}} = \frac{1}{\Re_1} + \frac{1}{\Re_2} + \cdots$

$e = \frac{d\lambda}{dt} = L\frac{di}{dt}, L = \frac{\lambda}{i} = \frac{N^2}{\Re}, \lambda = N\phi$

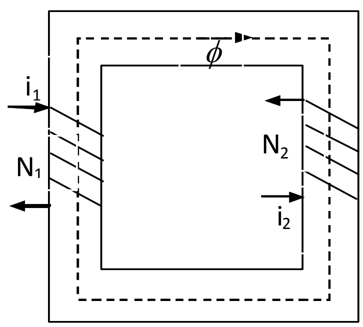


Figure R.2

$\lambda_1 = \lambda_{11} + \lambda_{12} = L_{11}i_1 + L_{12}i_2, \lambda_2 = \lambda_{21} + \lambda_{22} = L_{21}i_1 + L_{22}i_2$

$L_{11} = \frac{\lambda_{11}}{i_1}, L_{12} = \frac{\lambda_{12}}{i_2}, L_{21} = \frac{\lambda_{21}}{i_1} = L_{12}, L_{22} = \frac{\lambda_{22}}{i_2}$

Chapter 2. Electromechanical Energy Conversion

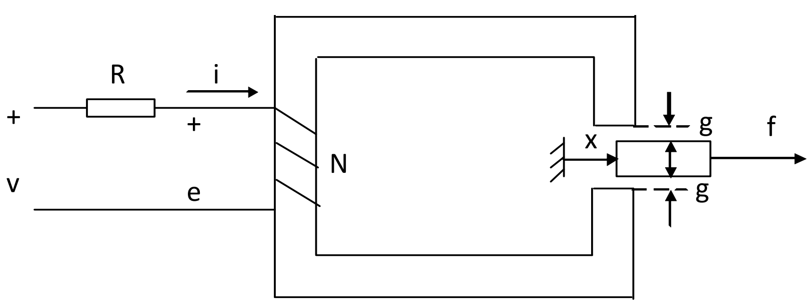


Figure R.3 A singly excited linear actuator

$W_\phi(\lambda, x) = \frac{1}{2L}\lambda^2, W_\phi(i, x) = \frac{1}{2}Li^2$

$f = -\frac{\partial W_\phi(\lambda,x)}{\partial x} = \frac{1}{2}i^2\frac{dL(x)}{dx}, f = \frac{\partial W_\phi(i,x)}{\partial x} = \frac{1}{2}i^2\frac{dL(x)}{dx}$

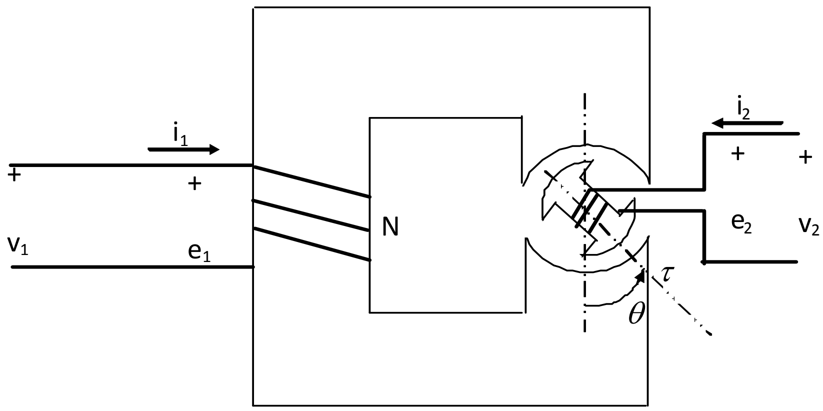


Figure R.4 A doubly excited actuator

$W_\phi(\lambda_1,\lambda_2,\theta) = \frac{1}{2}\Gamma_{11}^2\lambda_1^2 + \Gamma_{12}\lambda_1\lambda_2 + \frac{1}{2}\Gamma_{22}\lambda_2^2, W_\phi(i_1,i_2,\theta) = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1i_2 + \frac{1}{2}L_{22}i_2^2$

$\tau = -\frac{\partial W_\phi(\lambda_1,\lambda_2,\theta)}{\partial\theta} = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$

$\tau = \frac{\partial W_\phi(i_1,i_2,\theta)}{\partial\theta} = \frac{1}{2}i_1^2\frac{dL_{11}(\theta)}{d\theta} + i_1i_2\frac{dL_{12}(\theta)}{d\theta} + \frac{1}{2}i_2^2\frac{dL_{22}(\theta)}{d\theta}$

Chapter 3 Dynamics of Electromechanical Systems

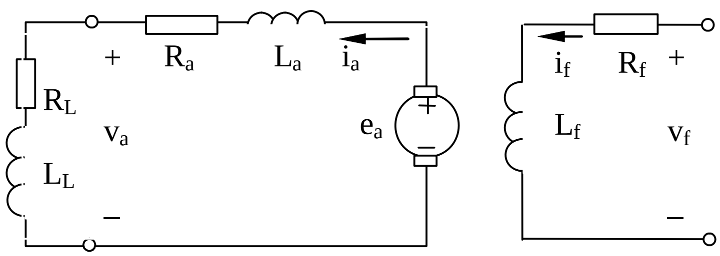


Figure R.5 Equivalent circuit of a dc generator

$V_f = R_fi_f(t) + L_f\frac{di_f(t)}{dt}$

$e_a(t) = K_e\omega i_f(t) = (R_a + R_L)i_a(t) + (L_a + L_L)\frac{di_a(t)}{dt}$

$I_f(s) = \frac{V_f(s)+L_fi_f(0)}{L_fs+R_f}$

$I_a(s) = \frac{K_e\omega(V_f(s)+L_fi_f(0))+(L_a+L_L)i_a(0)(L_fs+R_f)}{(L_fs+R_f)((L_a+L_L)s+R_a+R_L)}$

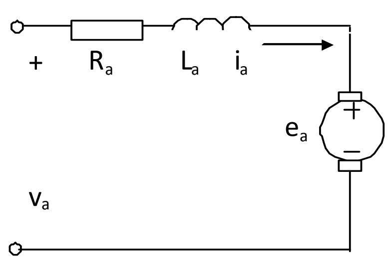


Figure R.6

$v_f(t) = R_fi_f(t) + L_f\frac{di_f(t)}{dt}$

$v_a(t) = R_ai_a(t) + L_a\frac{di_a(t)}{dt} + e_a(t)$

$e_a(t) = K_e i_f(t)\omega(t)$

$\tau_d(t) - \tau_L(t) - D\omega(t) = J\frac{d\omega(t)}{dt}$

$\tau_d(t) = K_\tau i_f(t) i_a(t)$

Chapter 4 Transformers

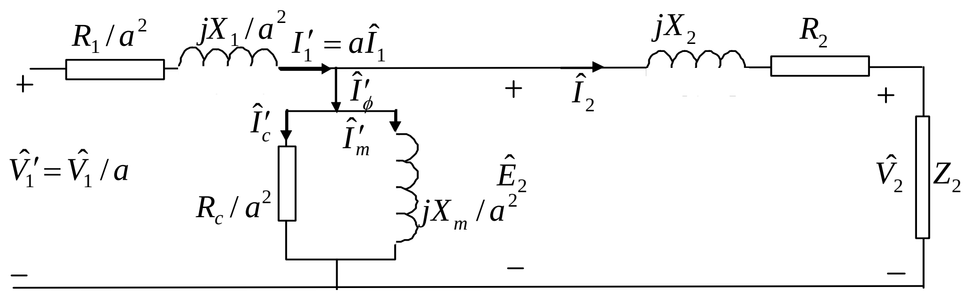


Figure R.7 The equivalent circuit as viewed from the secondary side

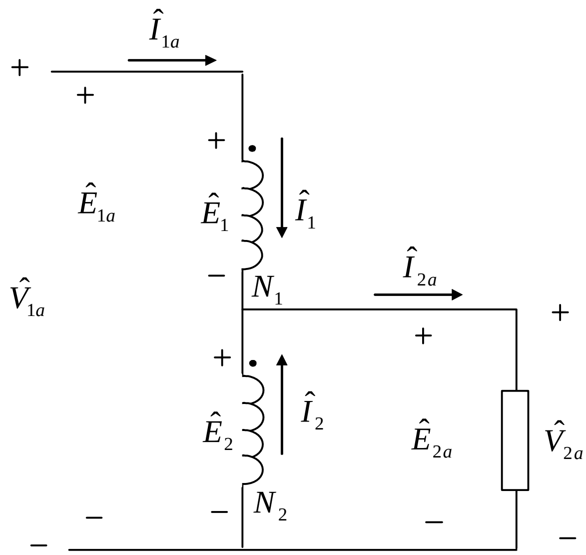


Figure R.8 A step-down authotransformer

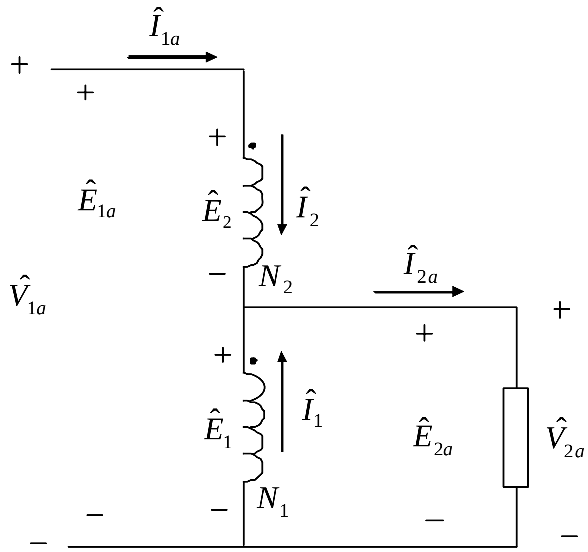


Figure R.9 A step-down authotransformer

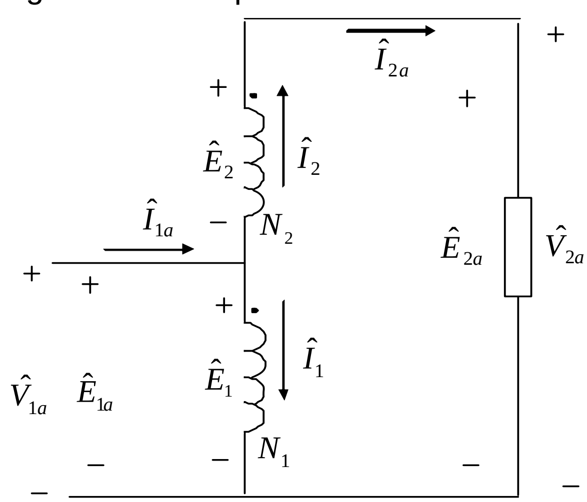


Figure R.10 A step-up autotransformer

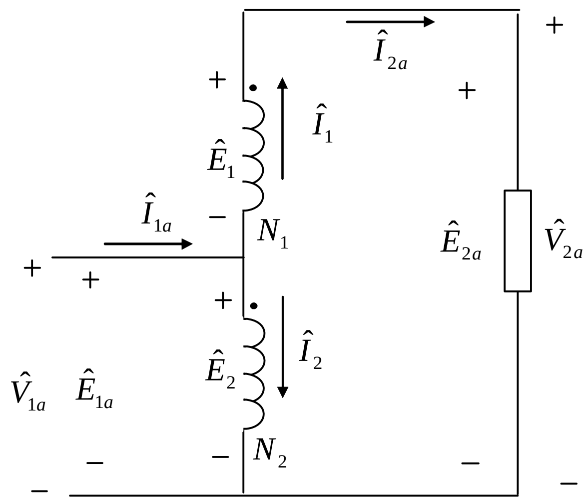


Figure R.11 A step-up autotransformer

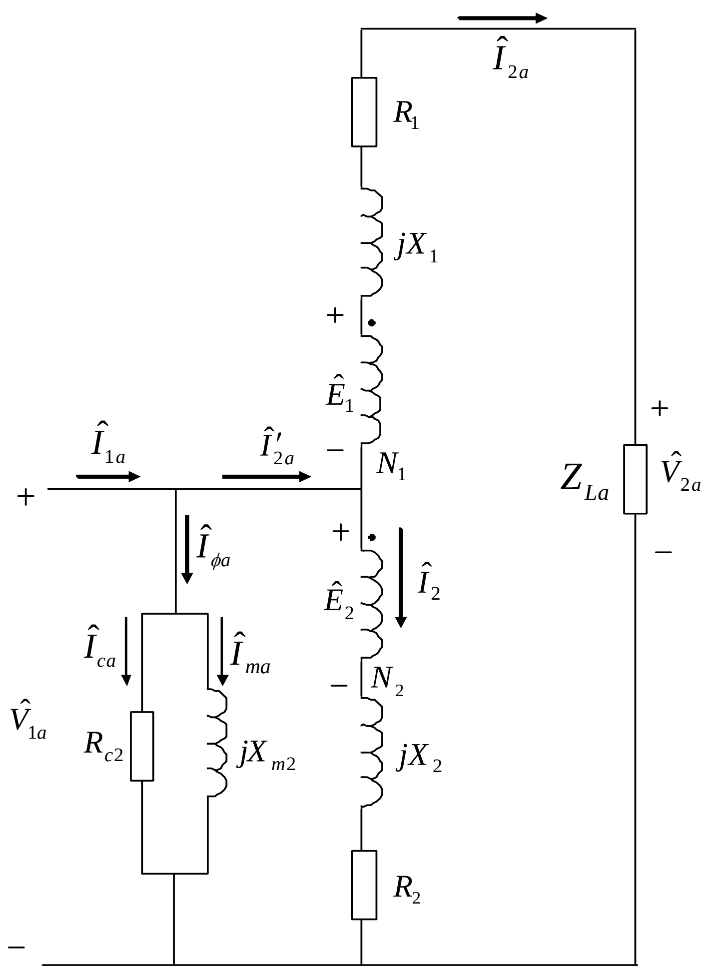
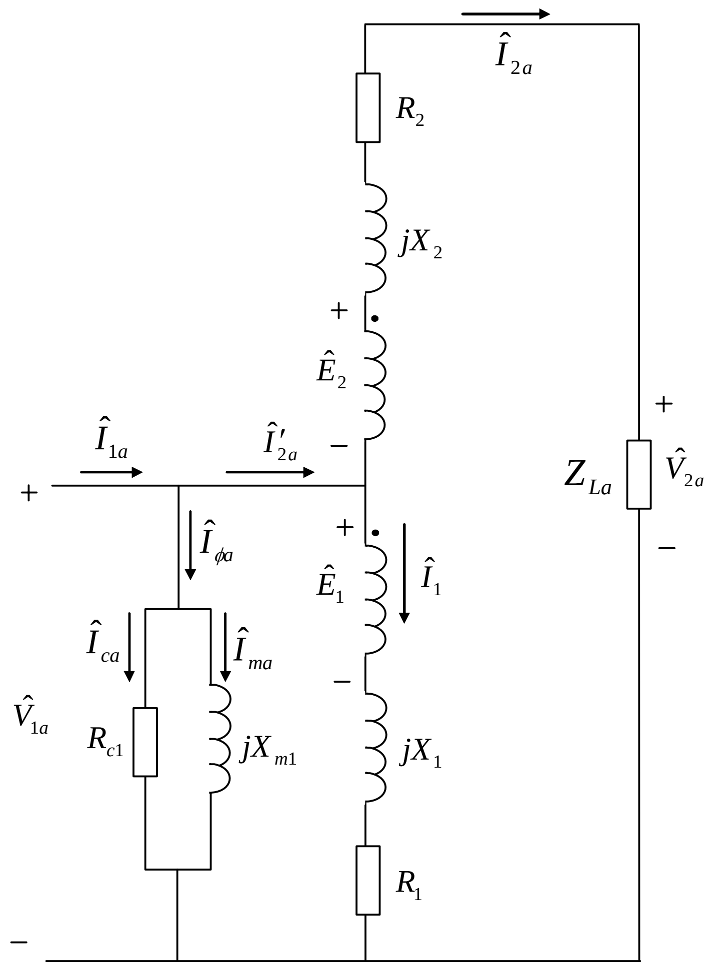


Figure R.12



$\hat{I}_{2a}$
R_2
jX_2
+
$\hat{E}_2$
−
$\hat{I}_{1a}$
$\hat{I}'_{2a}$
+
Z_{La}
$\hat{V}_{2a}$
−
+
$\hat{I}_{\phi a}$
+
$\hat{E}_1$
$\hat{I}_1$
−
$\hat{I}_{ca}$
$\hat{I}_{ma}$
$\hat{V}_{1a}$
R_{c1}
jX_{m1}
jX_1
R_1
−

Figure R.13

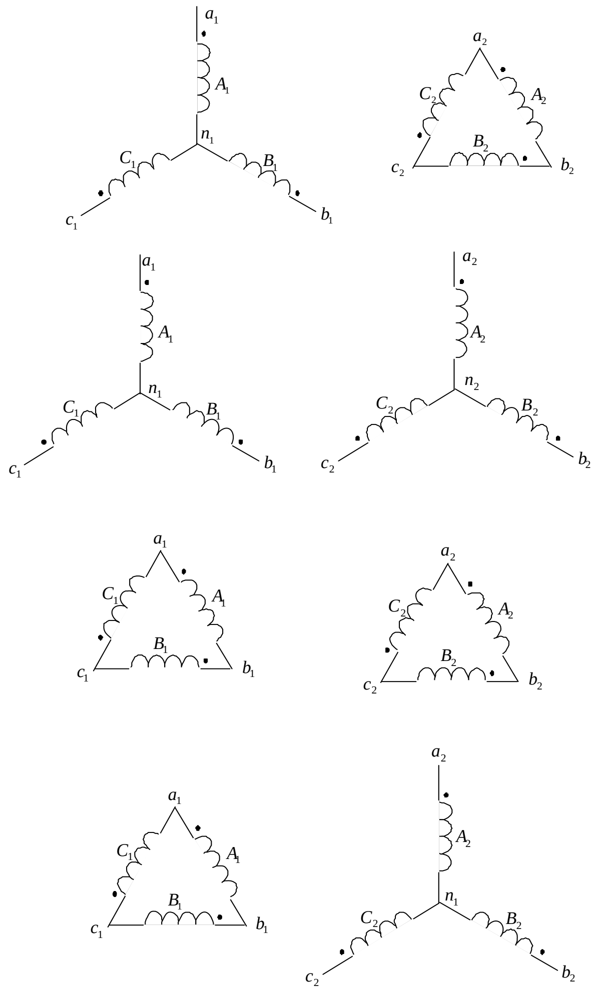


Figure R.14

$Z_Y = \frac{1}{3} Z_\triangle, \hat{V}_{an} = \frac{\hat{V}_{ab}}{\sqrt{3}} \angle -30^\circ, \hat{I}_{A'} = \sqrt{3}\, \hat{I}_A \angle -30^\circ$

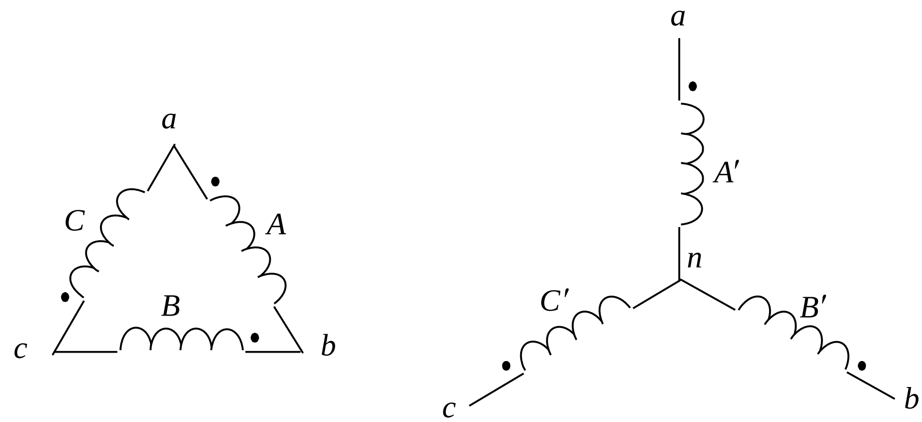


Figure R.18 $\triangle - Y$ transformation

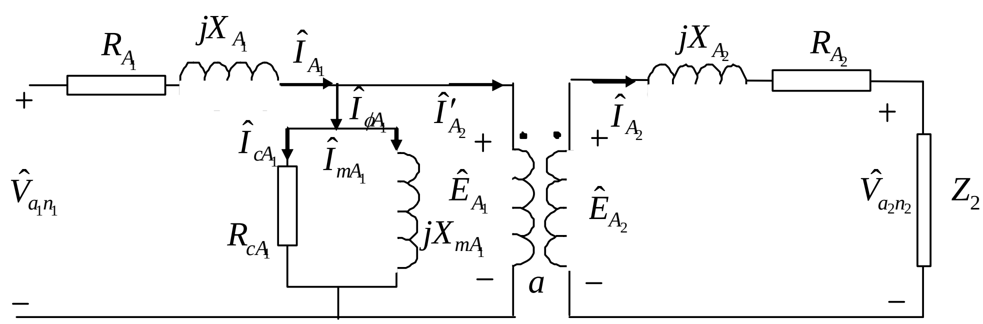


Figure R.19

$\triangle$-Y connection: $\hat{E}_{A_1} = a\hat{E}_{A_2}\angle -30^\circ$, $\hat{I}'_{A_2} = \frac{1}{a}\hat{I}_{A_2}\angle -30^\circ$

Y-$\triangle$ connection: $\hat{E}_{A_1} = a\hat{E}_{A_2}\angle 30^\circ$, $\hat{I}'_{A_2} = \frac{1}{a}\hat{I}_{A_2}\angle 30^\circ$

Chapter 5 Synchronous Machines

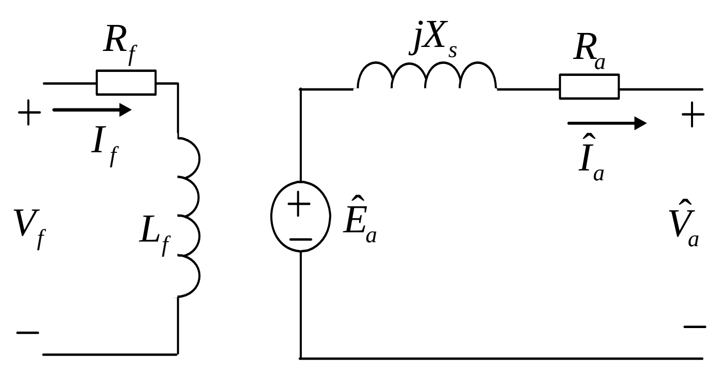


Figure R.20 The per-phase equivalent circuit of a

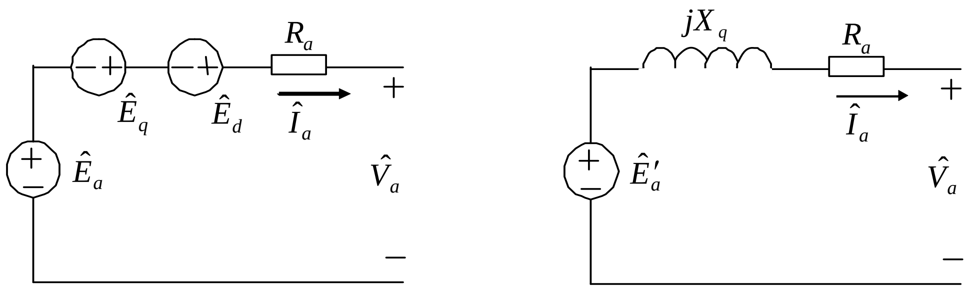


Figure R.21

$\hat{E}'_a = \hat{E}_a - j\hat{I}_d(X_d - X_q)$

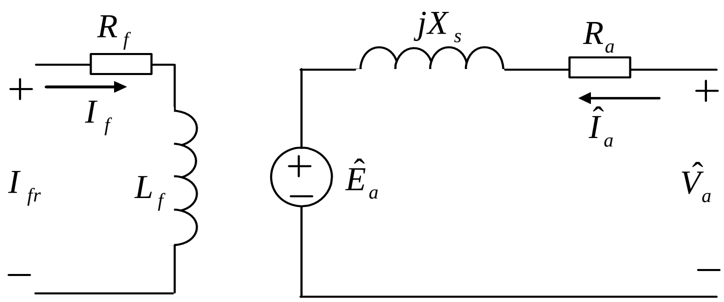


Figure R.22 The per-phase equivalent circiut of a

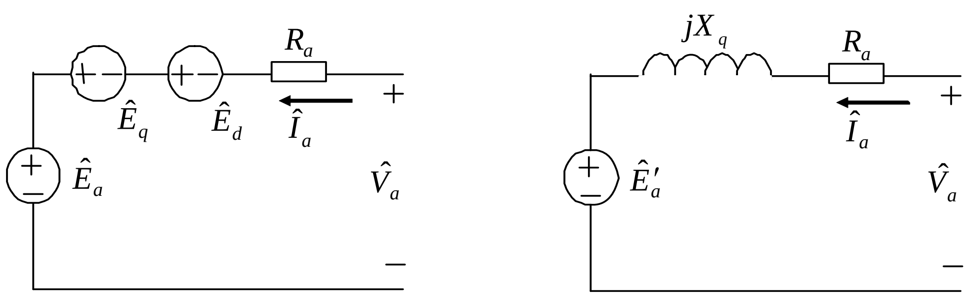


Figure R.23

$\hat{E}'_a = \hat{E}_a + j\hat{I}_d(X_d - X_q)$

Chapter 6 Induction Motors

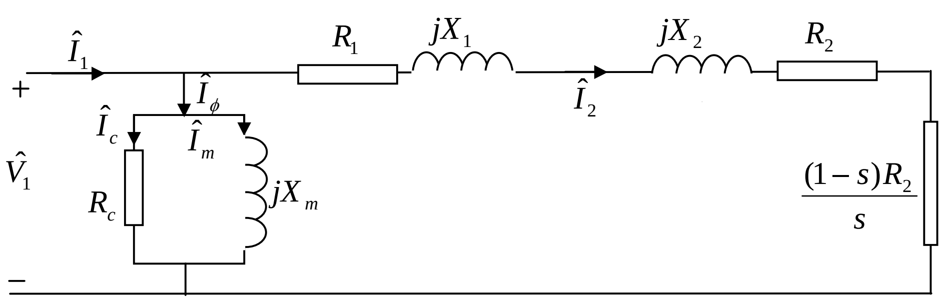


Figure R.24

$$P_d = 3I_2^2\frac{(1-s)R_2}{s} = \frac{3V_1^2\frac{(1-s)R_2}{s}}{\left(R_1+R_2+\frac{(1-s)R_2}{s}\right)^2+(X_1+X_2)^2}, \tau_d = \frac{3V_1^2R_2}{s\omega_s\left[\left(R_1+\frac{R_2}{s}\right)^2+(X_1+X_2)^2\right]}$$

$$s_{\max,p} = \frac{R_2}{R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}, P_{d,\max} = \frac{3}{2}\frac{V_1^2}{R_1+R_2+\sqrt{(R_1+R_2)^2+(X_1+X_2)^2}}$$

$$s_{\max,\tau} = \frac{R_2}{\sqrt{R_1^2+(X_1+X_2)^2}}, \tau_{d,\max} = \frac{3V_1^2}{2\omega_s\left[R_1+\sqrt{R_1^2+(X_1+X_2)^2}\right]}$$

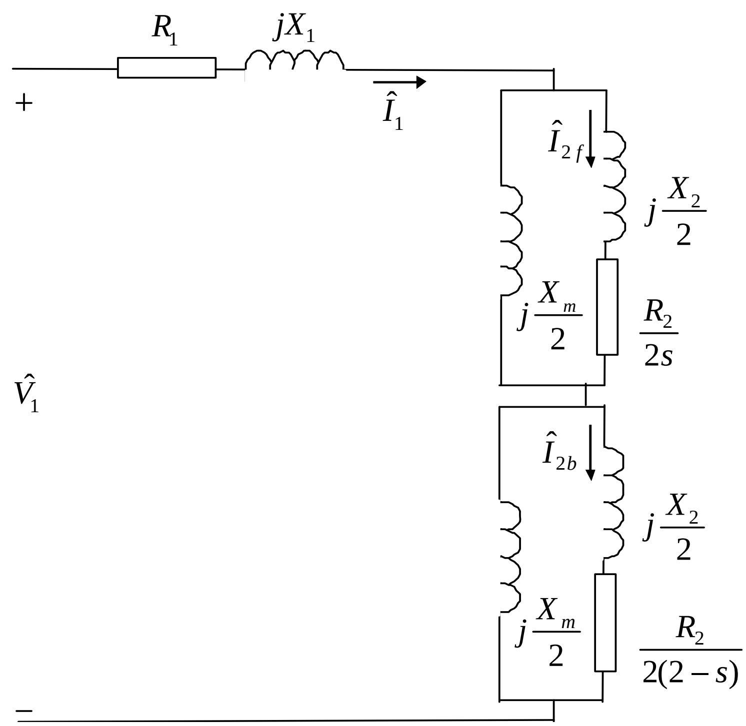


Figure R.25

$$Z_f = R_f + jX_f = 0.5\frac{jX_m(R_2/s+jX_2)}{R_2/s+j(X_2+X_m)} = 0.5\frac{X_m^2(R_2/s)}{(R_2/s)^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/s)^2+X_2(X_2+X_m)\right)}{(R_2/s)^2+(X_2+X_m)^2}$$

$$Z_b = R_b + jX_b = 0.5\frac{jX_m(R_2/(2-s)+jX_2)}{R_2/(2-s)+j(X_2+X_m)} = 0.5\frac{X_m^2(R_2/(2-s))}{(R_2/(2-s))^2+(X_2+X_m)^2} + j0.5\frac{X_m\left((R_2/(2-s))^2+X_2(X_2+X_m)\right)}{(R_2/(2-s))^2+(X_2+X_m)^2}$$

$$P_{agf} = I_1^2R_f = 0.5I_{2f}^2\frac{R_2}{s}, P_{agb} = I_1^2R_b = 0.5I_{2b}^2\frac{R_2}{2-s}$$

$$P_{df} = P_{agf} - P_{rcuf} = (1-s)P_{agf}, P_{db} = P_{agb} - P_{rcub} = -(1-s)P_{agb}$$

$$P_d = (1-s)P_{ag}, P_{ag} = P_{agf} - P_{agb}$$

$$P_d = (1-s)P_{ag} = \tau_d\omega_m = (1-s)\tau_d\omega_s$$

$$\tau_d = \frac{P_{agf}}{\omega_s} - \frac{P_{agb}}{\omega_s} = \tau_{fd} - \tau_{bd}$$

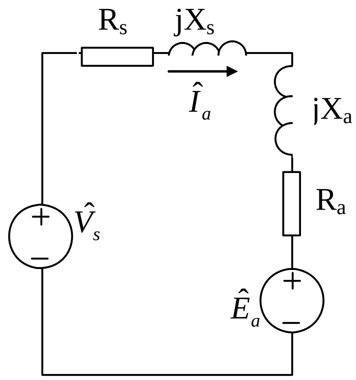


Figure R.26

$e_a(t) = K_a\Phi_a\omega(t), \tau_d(t) = K_a\Phi_a i_a(t)$

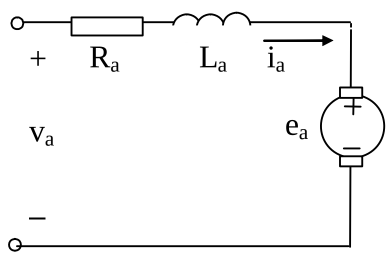


Figure R.27

$\theta_m = \frac{2}{P}\theta_e, n_m = \frac{1}{NP}n_{pulses}$