LC FOSTER NETWORKS

1. foster network 1 - partial fraction $Z(s)$

2. foster network 2 - partial fraction $Y(s)$

FOSTER 1

general expression for $Z(s)$ given by:

$$Z(s) = \frac{k(s^2 + \omega_{z_1}^2)(s^2 + \omega_{z_2}^2) \cdots (s^2 + \omega_{z_n}^2)}{s(s^2 + \omega_{p_1}^2)(s^2 + \omega_{p_2}^2) \cdots (s^2 + \omega_{p_n}^2)}$$

if the degree of N is highere then the degree of D, then we may perform division and we may write:

$$Z(s) = ks + Z_1(s)$$

Expand $Z_1(s)$ by partial fractions and combining complex conjugate pairs, we get:

$$Z(s) = ks + \frac{k_0}{s} + \frac{k_1 s}{(s^2 + \omega_{p_1}^2)} + \cdots + \frac{k_n s}{(s^2 + \omega_{p_n}^2)}$$

by combining complex conjugate terms, we mean:

$$\frac{\overline{k_{p_i}}}{(s - j\omega_{p_i})} + \frac{\overline{k_{p_i}}^*}{(s + j\omega_{p_i})} = \frac{(\overline{k_{p_i}} + \overline{k_{p_i}}^*)s + j\omega_{p_i}(\overline{k_{p_i}} - \overline{k_{p_i}}^*)}{(s^2 + \omega_{np_i}^2)}$$

$$= \frac{2k_{p_i} s}{(s^2 + \omega_{p_i}^2)} \quad , \quad \text{let} \quad k_i = 2k_{p_i} = \frac{k_i s}{(s^2 + \omega_{np_i}^2)}$$

$$Z(s) = ks + \frac{k_0}{s} + \frac{k_1 s}{(s^2 + \omega_{np_i^2})} + \dots + \frac{k_n s}{(s^2 + \omega_{pn^2})}$$

where all k's are positive and can be evaluated using the cover up technique.

$$k_0 = sZ(s) \Big|_{s=0}$$

$$k_i = \frac{(s^2 + \omega_{p_i^2})}{s} Z(s) \Big|_{s^2 = -\omega_{p_i^2}}, \quad i \in I$$

EX: Obtain the foster I realization given by

$$Z(s) = \frac{s(s^2 + 4)}{(s^2 + 1)(s^2 + 9)}$$

$$k = 0$$

$$k_0 = 0$$

$$Z(s) = \frac{k_1 s}{(s^2 + 1)} + \frac{k_2 s}{(s^2 + 9)}$$

$$k_1 = Z(s) \frac{(s^2 + 1)}{s} \Big|_{s^2 = -1} = \frac{s^2 + 4}{s^2 + 9} \Big|_{s^2 = -1} = \frac{3}{8}$$

$$k_2 = Z(s) \frac{(s^2 + 9)}{s} \Big|_{s^2 = -9} = \frac{s^2 + 4}{s^2 + 1} \Big|_{s^2 = -9} = \frac{5}{8}$$

$$Z(s) = Z_1(s) + Z_2(s) = \frac{\frac{3}{8}s}{s^2 + 1} + \frac{\frac{5}{8}s}{s^2 + 9}$$

$$Z_1(s) = $$

$$\frac{1}{Z_1} = sc + \frac{1}{sL} = \frac{s^2LC+1}{sL}$$

$$Z_1 = \frac{sL}{s^2LC+1} \quad \text{compared } /c \quad \frac{3/8\,s}{s^2+1}$$

$$\therefore \quad L = 3/8 \qquad LC = 1$$

$$\therefore \quad C = 8/3$$

$$Z_2(s) = $$

$$Z_2 = \frac{sL}{s^2LC+1} \quad \text{compared } \&\quad \frac{5/8\,s}{s^2+9}$$

$$L_2 = \frac{5}{72} \quad , \quad C_2 = \frac{8}{5}$$

FOSTER II NETWORK.

here we expand $Y(s)$ by partial fractions.
In most general form,

$$Y(s) = \frac{K(s^2 + \omega_{z_1}^2)(s^2 + \omega_{z_2}^2) \cdots (s^2 + \omega_{z_{n+1}}^2)}{s(s^2 + \omega_{p_1}^2)(s^2 + \omega_{p_2}^2) \cdots (s^2 + \omega_{pn}^2)}$$

which has a partial fraction expansion given by,

$$Y(s) = ks + \frac{k_0}{s} + \frac{k_1 s}{(s^2 + \omega_{p_1}^2)} + \dots + \frac{k_n s}{(s^2 + \omega_{pn}^2)}$$

All k's are positive real values, and can be found as follows,

$$k_0 = sY(s)\Big|_{s=0}$$

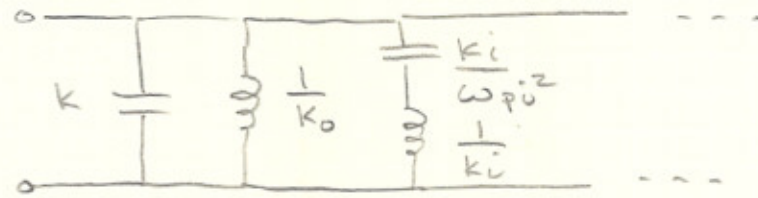$$k_i = \frac{(s^2 + \omega_{p_i}^2)}{s} Y(s)\Big|_{s^2 = \omega_{p_i}^2} \quad , \quad i \in I$$

$Y(s)$ respresents the sum of admitances, hence network realization contains structures in parrellel.

The first term is ks, which is the admitance of a capacitor. The second term k_0/s is the admitance of an inductor.

The third (etc) terms, are of the form

$$Y = \frac{k_i s}{(s^2 + \omega_{p_i}^2)} = \frac{1}{s\left(\frac{1}{k_i}\right) + \frac{1}{s\left(\frac{k_i}{\omega_{p_i}^2}\right)}} = \frac{1}{z}$$

Based on the above results, the complete realizati
is shown below.

EX: obtain the foster II realization for

$$Z(s) = \frac{s(s^2+4)}{(s^2+1)(s^2+9)}$$

$$Y(s) = \frac{(s^2+1)(s^2+9)}{s(s^2+4)}$$

$$= ks + \frac{k_o}{s} + \frac{k_1 s}{s^2+4}$$

$k = 1$

$$k_o = \left. \frac{(s^2+1)(s^2+9)}{(s^2+4)} \right|_{s=0} = \frac{9}{4} \ \Omega^{-1}$$

$$k_1 = \left. \frac{(s^2+1)(s^2+9)}{s^2} \right|_{s^2=-4} = \frac{15}{4}$$

$$Y = Y_1 + Y_2 + Y_3$$

$$Y_1 = s \quad ; \quad Y_2 = \frac{9}{4s} = \frac{4}{9} H$$

$$Y_3 = \frac{4}{15} H \quad \& \quad \frac{15}{16} H$$

RC FOSTER AND CAUER NETWORKS.

FOSTER I

$$Z(s) = \frac{k \{ (s + \alpha_2)(s + \alpha_4) \cdots \}}{\{ (s + \alpha_1)(s + \alpha_3) \cdots \}}$$
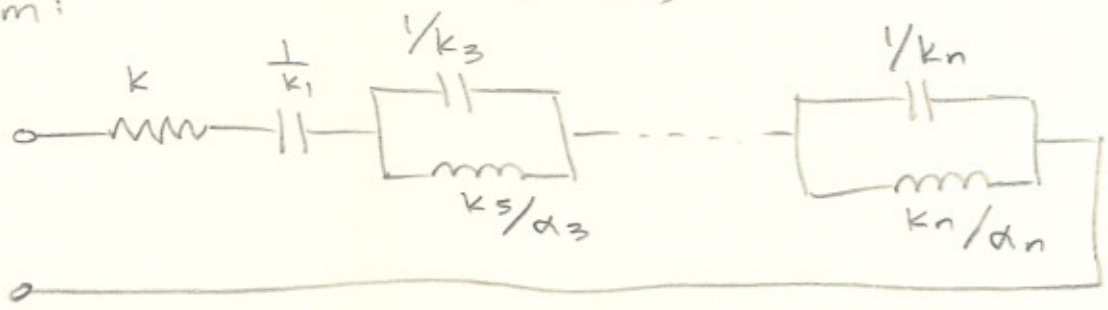
k is positive.

$$0 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \cdots$$

partial fractions give

$$Z(s) = k + \frac{k_1}{s} + \frac{k_3}{s + \alpha_3} + \frac{k_5}{s + \alpha_5}$$

k's are evaluated using coverup technique.

all k's are positive and real, ∴ RC foster I networks realization of $Z(s)$ will have the form:

FOSTER II

start with $Z(s)$ which is given by

$$Z(s) = \frac{k(s + \alpha_2)(s + \alpha_4) \cdots}{(s + \alpha_1)(s + \alpha_3) \cdots}$$
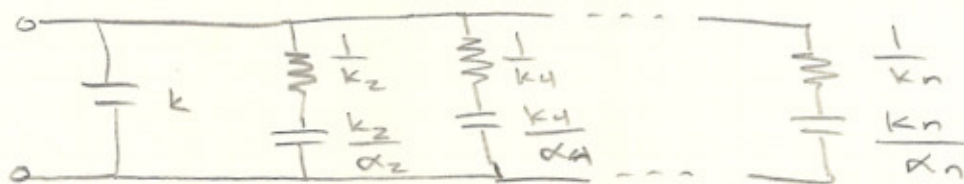
If $\alpha_1 = 0$ hence $k = \dfrac{1}{K}$

$$Y(s) = \frac{ks(s+\alpha_3)(s+\alpha_5)\dots}{(s+\alpha_2)(s+\alpha_4)\dots}$$

$$\frac{Y(s)}{s} = \frac{k(s+\alpha_3)(s+\alpha_5)\dots}{(s+\alpha_2)(s+\alpha_4)\dots}$$

partial fractions.

$$\frac{Y(s)}{s} = k + \frac{k_2}{(s+\alpha_2)} + \frac{k_4}{(s+\alpha_4)} + \dots + \frac{k_n}{(s+\alpha_n)}$$

foster II realization.

EX: given

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)(s+4)}$$

realize foster I & II networks, and also realize Cauer I & II

2nd ORDER SYSTEMS

most general 2nd order functions are known as the "biquad" functions as shown below.

$$T(s) = \frac{C_2 s^2 + C_1 s + C_0}{d_2 s^2 + d_1 s + d_0}$$

$$T(s) = \frac{\frac{C_2}{d_2} s^2 + \frac{C_1}{d_2} s + \frac{C_0}{d_2}}{s^2 + \frac{d_1}{d_2} s + \frac{d_0}{d_2}}$$

and we write

$$T(s) = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + b_1 s + b_0}$$

$$T(s) = \frac{N(s)}{D(s)}$$

when poles are properly placed $N(s)$ can be adjusted to make

$T(s)$ low pass $(a_2 = a_1 = 0)$

$T(s)$ band pase $(a_2 = a_0 = 0)$

$T(s)$ high pass $(a_1 = a_0 = 0)$

$T(s)$ all pass $(a_2 = 1, a_1 = -b_1, a_0 = b_0)$

$T(s)$ band stop $(a_1 = 0)$

The above designations refer to $|T(s)| = |T(j\omega)|$ = magnitude of transfer function

The above designations of $|T(\omega j)|$ indicate how the amplitude of sine waves with different freq are affected in the steady state as they are processed by the network.

For example, low pass function passes low pass function passes low freq and rejects high freq.

$$\text{let: } D(s) = (s+\alpha)^2 + \beta^2$$

$$= s^2 + 2\alpha s + \alpha^2 + \beta^2$$

$$\text{let } \omega_0^2 = \alpha^2 + \beta^2$$

$$\text{let } 2\alpha = \frac{\omega_0}{Q}$$

$$\therefore \quad D(s) = s^2 + \frac{\omega_0}{Q}s + \omega_0^2$$

where

$$\omega_0 = \sqrt{\alpha^2 + \beta^2}$$

$$Q = \frac{\sqrt{\alpha^2 + \beta^2}}{2\alpha}$$

If ω_0 and Q are given, we can find α and β from:

$$\alpha = \frac{\omega_0}{2Q}$$
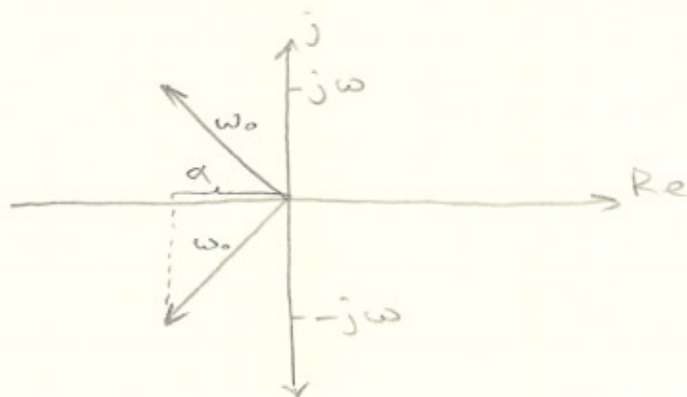
$$\beta = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}$$

from $D(s) = (s+\alpha)^2 + \beta^2$

if we evaluate $D(s) = 0$

$$(s+\alpha)^2 + \beta^2 = 0$$

$$s^2 + 2\alpha s + \alpha^2 + \beta^2 = 0$$

$$s = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4(\alpha^2 + \beta^2)}}{2}$$

$$s = -\alpha \pm j\beta$$

w_o: is the magnitude of the poles, distance from the origin to the poles.

Q: is the measure of the slope of the radial line from the origin to the pole. The higher the Q the steeper the radial lines.

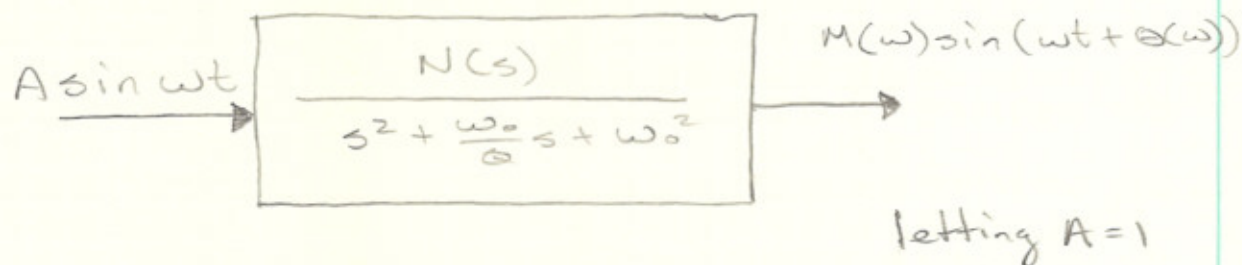when $Q \leq \frac{1}{2}$, roots are real. Let us check this out.

$$D(s) = s^2 + \frac{w_o}{Q} s + w_o^2$$

$$s = \frac{-w_o/Q \pm \sqrt{(w_o/Q)^2 - 4w_o^2}}{2}$$

$$s = \frac{\omega_0/(1/2) \pm \sqrt{(\omega_0/(1/2))^2 - 4\omega_0^2}}{2}$$

$$= -\frac{2\omega_0}{2} = -\omega_0$$

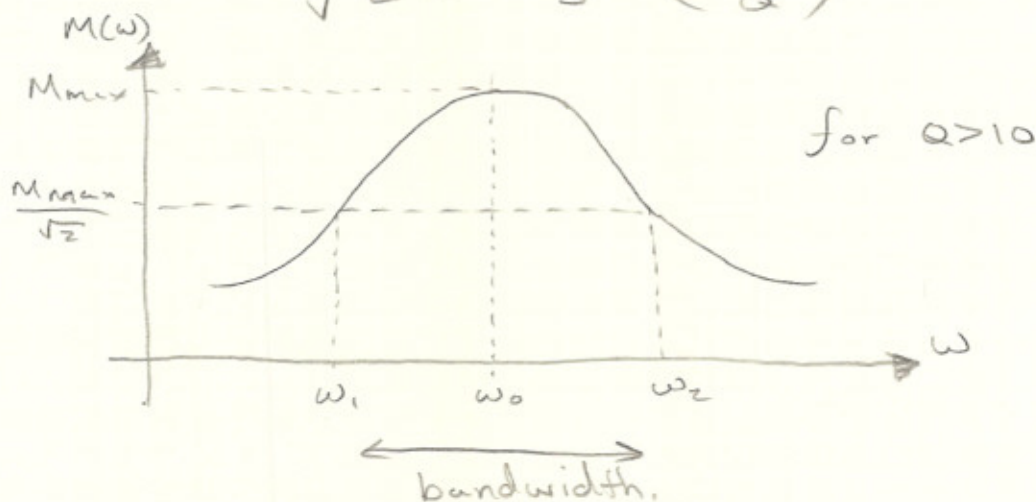Consider a sinusodial steady state responce given below.

$$A\sin \omega t \longrightarrow \boxed{\frac{N(s)}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}} \longrightarrow M(\omega)\sin(\omega t + \alpha(\omega))$$

letting $A = 1$

The above is represented by:

$$\frac{M(\omega)}{A} = |T(j\omega)| = \left| \frac{N(s)}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \right|$$

remember that $s = j\omega$

$$M(\omega) = \frac{|N(j\omega)|}{\sqrt{[\omega_0^2 - \omega^2]^2 + \left(\frac{\omega_0 \omega}{Q}\right)^2}}$$

for $Q > 10$

bandwidth.

note: for steady state responce.

Note: for all practical purposes $M(\omega)$ verus ω peaks at $\omega = \omega_0$.

The two freqs at either side of ω_0, where the magnitudes

$$M(\omega) = \frac{[M(\omega)]_{max}}{\sqrt{2}}$$

this is also referred to as the 3dB frequencies.

DECIBELS

Is the logarithmic measurement of the ratio of one power to another or one voltage to another.

$$dB = 10 \log \left(\frac{P_{out}}{P_{in}} \right)$$

$$dB = 20 \log \left(\frac{V_{out}}{V_{in}} \right)$$

$$V_{out} = \frac{V_{in}}{\sqrt{2}} = 0.707 V_{in}$$

$$\frac{V_{out}}{V_{in}} = 0.707$$

$$\therefore \quad 20 \log (0.707) = -3 \, dB.$$

Q

the higher the value of Q, the narrower the bandwidth, since

$$BW = \frac{\omega_0}{Q},$$

and the more selective the system is for sinusoidal signals. High Q indicates that the poles are closer to the imaginary axis

SUMMARY (biquad functions)

$$TF = k \left\{ \frac{a_2 s^2 + a_1 \frac{w_0}{Q} s + a_0 w_0^2}{s^2 + \frac{w_0}{Q} s + w_0^2} \right\}$$

the above is the most common biquad functions used in a variety of filter designs.

$k =$ d.c. gain.

$a_2, a_1,$ and a_0 are constants in we let $s = jw$

let $k = 1$

$$T(jw) = \frac{(a_0 w_0^2 - a_2 w^2) + j a_1 \left(\frac{w_0}{Q}\right) w}{(w_0^2 - w^2) + j \left(\frac{w_0}{Q}\right) w}$$

let $w = 2\pi f$

It can be shown that:

$$Q = \frac{w_0}{BW} = \frac{w_0}{w_2 - w_1}$$

$w_2 :$ high cut off freq.

$w_1 :$ low cut off freq.

different values for a_2, a_1, a_0 yeild to

Filter	a_2	a_1	a_0	TF.
LP	0	0	1	$\dfrac{k\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$
HP	1	0	0	$\dfrac{ks^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$
BP	0	1	0	$\dfrac{k\left(\frac{\omega_0}{Q}\right)s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$
Band reject	1	0	1	$\dfrac{k(s^2 + \omega_0^2)}{s^2 + \left(\frac{\omega_0}{Q}\right)s + \omega_0^2}$
All Pass	1	-1	1	$\dfrac{k\left(s^2 - \frac{\omega_0}{Q}s + \omega_0^2\right)}{s^2 + \left(\frac{\omega_0}{Q}\right)s + \omega_0^2}$

ROOT LOCI AND SENSITIVITY FUNCTIONS. 2ND ORDER.
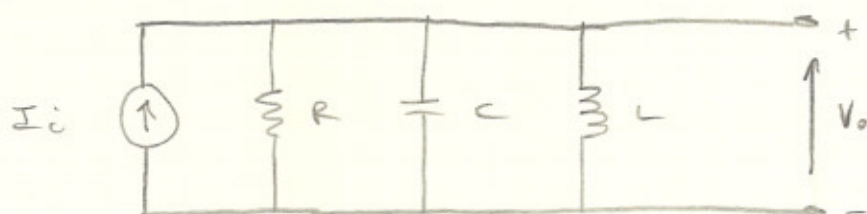
a system transfer function for

$$T(s) = \frac{V_o}{V_i}(s)$$

relates the output to the input variable.

Physical electrical systems. in general depart from desired system characteristics for many reasons, for example, Capacitor and resistors are not constants, and their values change with temp, age, etc. and the above include inductors In practical circuits, capacitors & resistor values. change by as much as ±5% from their exact values.

So in real life a system which is designed with a certain pole-zero distribution to achieve a specified freq responce may exhibit different characteristics when built and tested. So it is important to consider the impact of changes in component responce.

Consider the following:

$$TF = \left(\frac{V_o}{I_i}\right)(s) = \frac{s}{\frac{1}{C}\left\{s^2 + \frac{1}{RC}s + \frac{1}{LC}\right\}}$$

here, $w_o = \frac{1}{\sqrt{LC}}$, $Q = R\sqrt{\frac{C}{L}}$
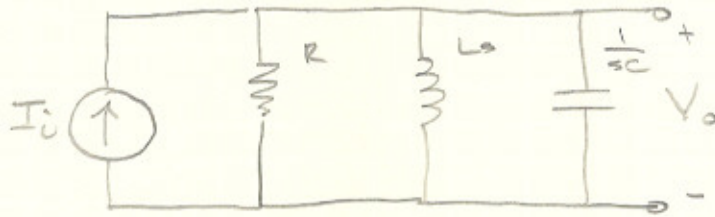
Comparing to earlier work, we see that

$$\alpha = \frac{1}{2RC}$$

$$\beta = \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$$

from the above we notice that wo depends on L and C and not of R

α is independant of L, so we can vary Q of the pole and keep wo constant by changing by changing R.

consider the following circuit

$$TF = \frac{V_o}{I_i} = \frac{s/c}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

∴ poles are

$$s = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2}$$

$$s = \frac{-1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}$$

case 1

$$\left(\frac{1}{RC}\right)^2 = \frac{4}{LC} \Rightarrow c = \frac{L}{4R^2}$$

hence

$$s = \frac{-1}{2RC} = \frac{-2R}{L}$$

$$\boxed{c = \frac{L}{4R^2} \quad , \quad s = \frac{-2R}{L}}$$

case 2:

$$D(s) = s^2 + \frac{s}{RC} + \frac{1}{LC} = 0 \quad \text{given roots}$$

$$s^2 LR + s\frac{L}{C} + \frac{R}{C} = 0$$

when $C \to \infty$, $s^2 \to 0$

when $C \to 0$, $s \to -\frac{R}{L}$

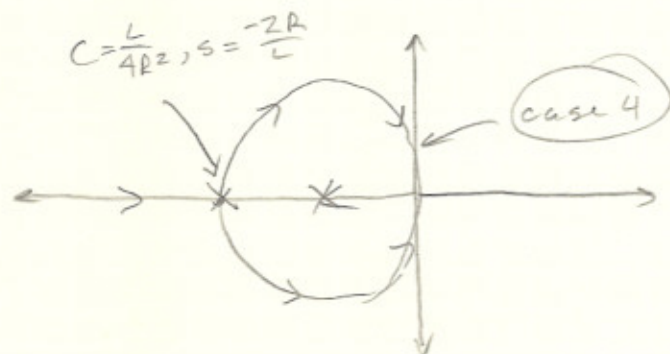case 3: $\left(\frac{1}{RC}\right)^2 > \frac{4}{LC}$

$$s = \frac{-1}{2RC} \pm \frac{1}{2}\underbrace{\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}$$

results in a real
number.

hence we have poles on the real axis.

case 4 $\left(\frac{1}{RC}\right)^2 < \frac{4}{LC}$

$$s = \frac{-1}{2RC} + \frac{1}{2}j\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}$$

$C = \frac{L}{4R^2}, s = \frac{-2R}{L}$

SENSITIVITY

The responce of a practical network is likely to deviate from that predicted by theory. This is due to the component theorances and quite often operational amp. Aging my aslo plan a role.

It is important then to show how the network is sensitive to component variations. For example, a filter designer may want to know the extent to which a 1% variation in a given resistance or capaciterce may effect W_o and BW.

Given a filter parameter y such as W_o and Q, a given filter component x such as resistance R, or capacitance C, the sensitivity function is defined as:

$$S_x^y = \frac{\partial y / y}{\partial x / x} = \frac{x}{y} \frac{\partial y}{\partial x}$$

The partial derivatives are used, since the filter parameter depends on more than just one component.

The sensitivity function satisfies the following useful properties.

$$S_{\frac{1}{x}}^y = S_x^{1/y} = -S_x^y \qquad \text{(A)}$$
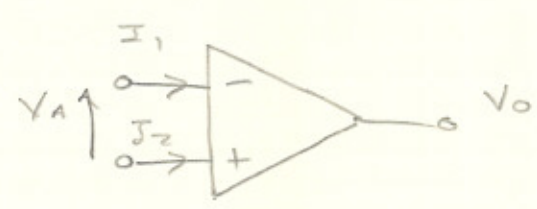
$$S_x^{y_1 y_2} = S_x^{y_1} + S_x^{y_2} \qquad \text{(B)}$$

$$S_x^{y_1 / y_2} = S_x^{y_1} - S_x^{y_2} \qquad \text{(C)}$$

$$S_x^{x^n} = n \qquad \text{(D)}$$
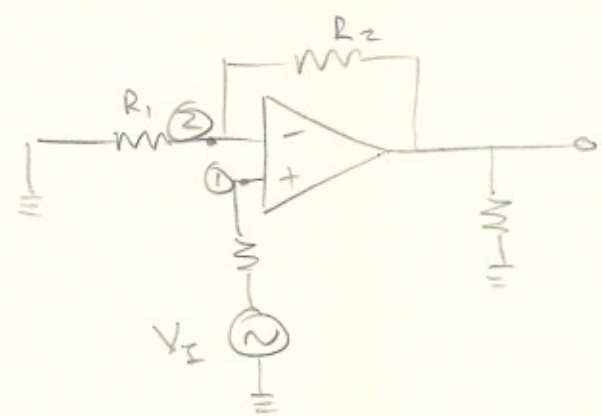
$$S_{x_1}^y = S_{x_2}^y \, S_{x_1}^{x_2} \qquad \text{(E)}$$

THE OP AMP.

An ideal op amp will have a very high gain and the symbol of the op amp is shown below.

$$V_A \uparrow \quad I_1 \qquad V_o$$
$$I_2$$

Ideal op amp

1. ∞ input impedance.

2. $I_1 = I_2 = 0$

3. $\emptyset$ output impedance V_o is independent of current drawn.
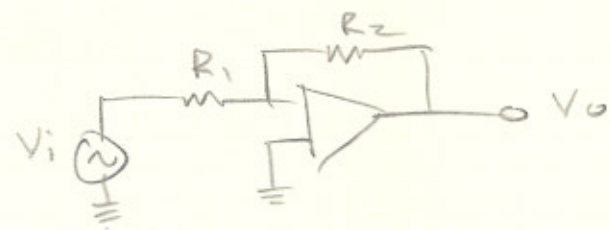
NON INVERTING AMP

$$V_① = V_②$$

$$\frac{V_o}{V_i} = 1 + \frac{R_2}{R_1}$$

$$\frac{V_o}{V_i}(s) = 1 + \frac{Z_2(s)}{Z_1(s)}$$

INVERTING AMP.

$$\frac{V_o}{V_i} = -\frac{R_2}{R_1}$$

$$\frac{V_o}{V_i}(s) = -\frac{Z_2(s)}{Z_1(s)}.$$

DIFFERENCE AMP