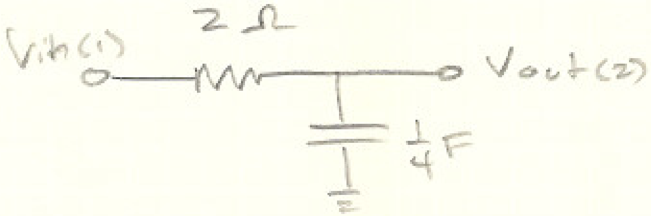
EX. Determine the TF of the circuit shown below.

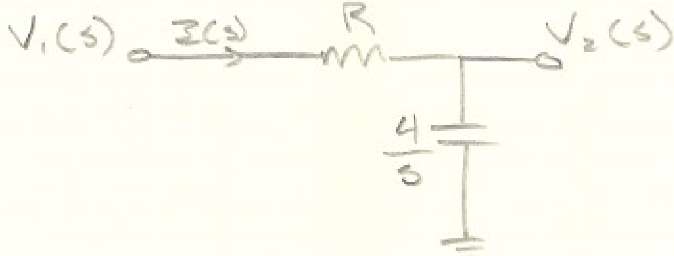


use the transfer function to solve for

A. $v_2(t)$, when $v_1(t)$ is a unit impulse function

B. $v_2(t)$, when $v_1(t) = 5\sin 2t$ (V)

SOL. A. Transform the above circuit to the s-domain.



$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{4/s}{2 + 4/s} = \frac{2}{s+2}$$

$V_1(s) = 1$

$$\therefore V_2(s) = \frac{2}{s+2} V_1(s)$$

$$v_2(t) = \mathcal{L}^{-1}\left[\frac{2}{s+2}\right] = 2e^{-2t}$$

SOL. B. $v_1(t) = 5\sin 2t$

$$V_1(s) = \frac{5(2)}{s^2+4} = \frac{10}{s^2+4}$$

$$V_2(s) = G(s)V_1(s)$$
$$= \frac{2}{s+2} \cdot \frac{10}{s^2+4}$$
$$= \frac{20}{(s+2)(s^2+4)}$$

H/w, verify that

$$v_2(t) = \frac{5}{2}e^{-2t} - \frac{5}{2}\cos 2t + \frac{5}{2}\sin 2t$$

let

$$-\frac{5}{2}\cos 2t + \frac{5}{2}\sin 2t = E\sin(2t+\phi)$$

$$\left.\begin{aligned} -\frac{5}{2} &= E\sin\phi \\ +\frac{5}{2} &= E\cos\phi \end{aligned}\right\} \begin{aligned} &\tan\phi = -1 \\ &\phi = -45 \\ &E = \frac{\sqrt{2}\,5}{2} \end{aligned}$$

final answer

$$v_2(t) = \frac{5}{2}e^{-2t} + \frac{5\sqrt{2}}{2}\sin(2t - 45^\circ)$$

note: solve the same problem using phasors.

$$V_1 = \left(2 + \frac{1}{j\omega C}\right)I \quad , \quad I = j\omega C V_2$$

$$\therefore V_1 = \frac{2j\omega C + 1}{j\omega C}\left[j\omega C\right]V_2$$

$$V_1 = (2j\omega C + 1)V_2$$

which we can write as.

$$\frac{V_2}{V_1} = \frac{1}{2j\omega c+1} = \frac{1}{2(j)(\frac{1}{4})(2)+1} = \frac{1}{j+1}$$

but $v(t) = 5\sin 2t \Rightarrow V_1 = 5\angle 0°$

$$\therefore \quad V_2 = \frac{5\angle 0°}{1+j}$$

$$= \frac{5\angle 0°}{\sqrt{1^2+1^2}\angle 45°}$$

$$= \frac{5}{\sqrt{2}}\angle -45°$$

$$\therefore \quad V_2 = \frac{5}{\sqrt{2}}\sin(2t-45°)$$

now here you should see that phasors are a subset from laplace transform. And will only give you the steady state.

LAPLACE TRANSFORM OF UNIT STEP FUNCTION.

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

Theorem:

If $F(s) = \mathcal{L} f(t)$

and $f(t-a)u(t-a) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \end{cases}$

$$\mathcal{L}\left[f(t-a)u(t-a)\right] = e^{-as}F(s)$$

and vica versa.

DEFINITIONS OF LINEAR SYSTEMS.

a system is said to be linear with respect to an excitation $x(t)$ and a responce $y(t)$ if the following two properties are satisified:

PROPERTY 1 (amplitude property)

If the excitation $x(t)$ produces a responce $y(t)$, then the excitation $Kx(t)$ should produce a responce $Ky(t)$, where K is constant

EX. an input of 10V produces an output current of 2 A, then an input of 25V should produce a current of 5 A.

PROPERTY 2 (super position principle)

If an excitation of $x_1(t)$ produces a responce of $y_1(t)$ and an excitation of $x_2(t)$ produces $y_2(t)$ then:

an excitation of $x_1(t) + x_2(t)$ should produce a responce of $y_1(t) + y_2(t)$

EX. 10 units produce an output of 20t and an input of 5cost produces 10sint then

$10 + 5\cos t$ produces $20 + 10\sin t$

TRANSFER FUNCTIONS.

a general standard lumped system with an excitation $x(t)$ and a responce of $y(t)$ can be described by an most general differential equation given by:

(A) $b_m \frac{d^m y}{dt^m} + b_{m-1}\frac{d^{m-1} y}{dt^{m-1}} + \ldots + b_0 y = a_n \frac{d^n x}{dt^n} + \ldots + a_0 x$

highest derivative in y determines the order of the system. So the above equation describes an n^{th} order system. Futther more, we will require $m \geq n$

The next is to take the laplace transform of both sides, then we will get a rather complicated terms with respect to the initial conditions

(A) Transformed will give:

$$\left[b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0\right] Y(s) + I_y =$$

$$= \left[a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right] X(s) + I_x$$

where I_x & I_y are the mass collection of initial conditions.

The last equation can be simplified to.

$$Y(s) = \left[\frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}\right] X(s) + \frac{I_x - I_y}{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}$$

If the initial conditions are zero, then the equation becomes.

$$Y(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0} X(s)$$

or $$\frac{Y(s)}{X(s)} = G(s) = T.F.$$

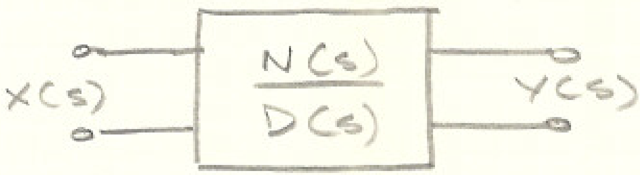
For convience, wrte

$$N(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0$$

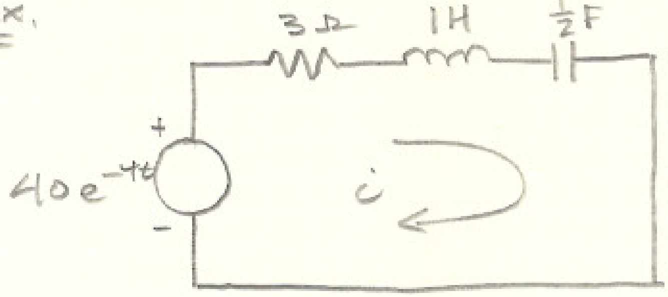
$$D(s) = b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0$$

then

$$G(s) = \frac{N(s)}{D(s)}$$

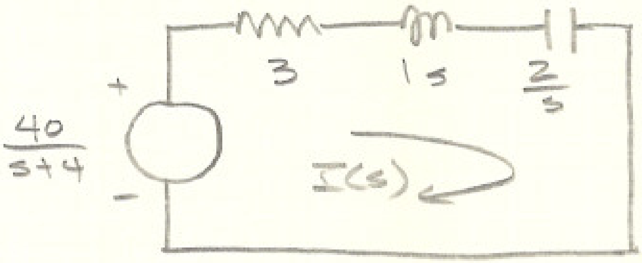


Ex.



Assume that there is no initial energy stored in the circuit.

SOL The transformed circuit.



$$Z(s) = 3 + s + \frac{2}{s} = \frac{s^2 + 3s + 2}{s}$$

then we write

$$V(s) = Z(s)\,I(s)$$

or $$I(s) = \frac{V(s)}{Z(s)}$$

$$\text{T.F.} = \frac{1}{Z(s)} = Y(s)$$

admittance

impedance

$$I(s) = \frac{s}{s^2+3s+2}\left(\frac{40}{s+4}\right)$$

$$I(s) = \frac{40s}{(s^2+3s+2)(s+4)}$$

$$I(s) = \frac{40s}{(s+2)(s+1)(s+4)}$$

partial fractions, then

$$i(t) = \mathcal{L}^{-1}\left[\frac{40}{s+2} + \frac{\left(-\frac{40}{3}\right)}{s+1} + \frac{\left(-\frac{80}{3}\right)}{s+4}\right]$$

$$i(t) = \left[40e^{-2t} - \frac{40}{3}e^{-t} - \frac{80}{3}e^{-4t}\right]u(t)$$

POLES, ZEROS, AND THEIR EFFECT ON NETWORK RESPONCE.

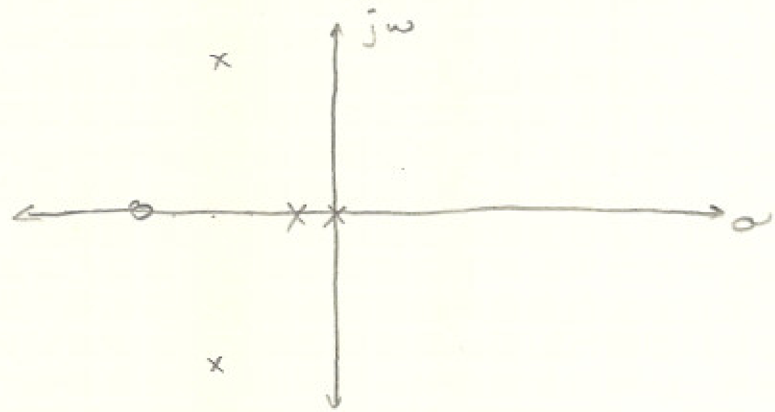
consider

$$G(s) = \frac{20(s+5)}{s(s+1)(s^2+6s+25)}$$

analysising

A. $G(s)$ has a zero at -5, which will be represented by $\emptyset$

B. $G(s)$ has poles at $s=0$, $s=-1$, and $s=-3 \pm j4$.

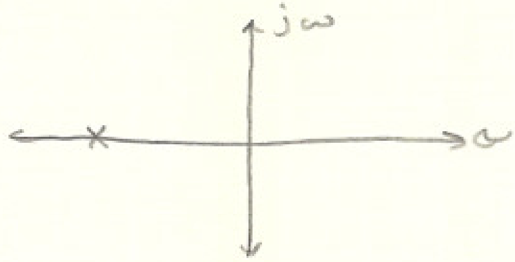


CASES (T.F.)

1. REAL ROOTS

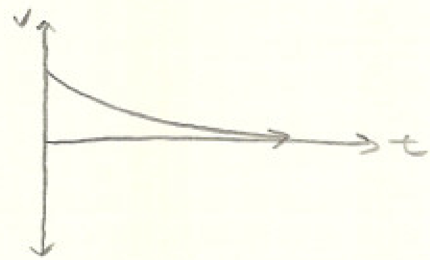
$$V_1 \rightarrow \boxed{G(s)} \rightarrow V_2$$

A. $\frac{V_2}{V_1}(s) = G(s) = \frac{1}{s+4}$

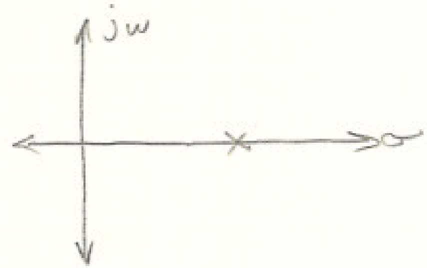


if $V_1(t)$ is a unit implose function then $V_1(s) = 1$

$\therefore V_2(s) = \frac{1}{s+4} = e^{-4t}$

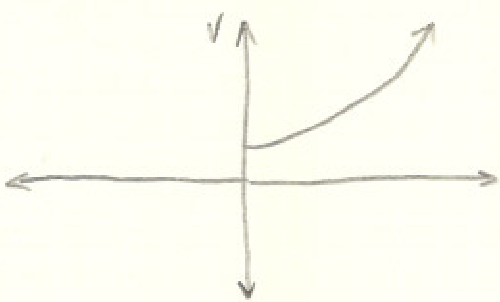


B. $\frac{V_2}{V_1}(s) = G(s) = \frac{1}{s-4}$



when $V_1(s) = 1$ then $V_2(s) = \frac{1}{s-4}$

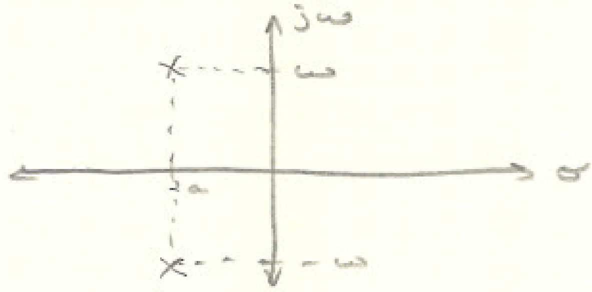
and $v_2(t) = e^{4t}$

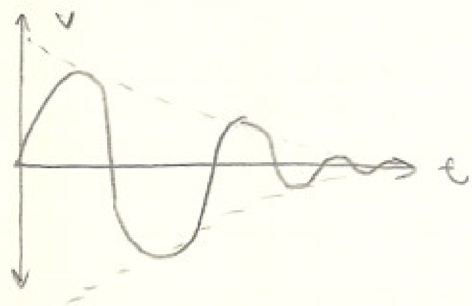


2. COMPLEX ROOTS

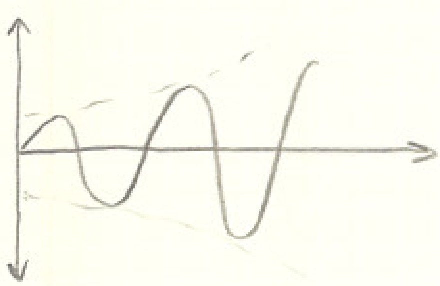
$$\frac{V_2}{V_1}(s) = \frac{\omega}{(s+a)^2+\omega^2} = G(s)$$

if $V_1(s) = 1$ then $v_2(t) = e^{-at}\sin\omega t$





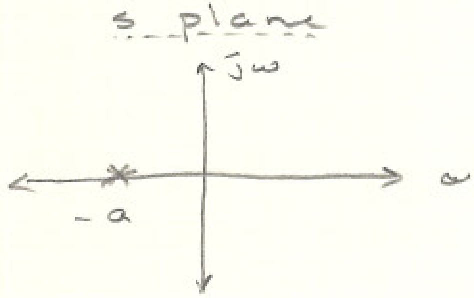
if we had $G(s) = \frac{\omega}{(s-a)^2+\omega^2}$



blow up.

DOUBLE POLES.

- $$\frac{V_2}{V_1} = \frac{1}{(s+a)^2}$$

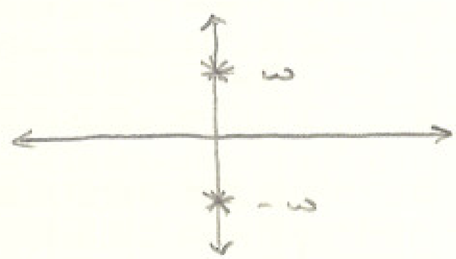


If $V_1(s) = 1$

then $\mathcal{L}\, V_2(s) = v_2(t) = te^{-at}$

- double conjugate poles

$$\frac{V_2}{V_1}(s) = \frac{s}{(s^2+\omega^2)^2}$$

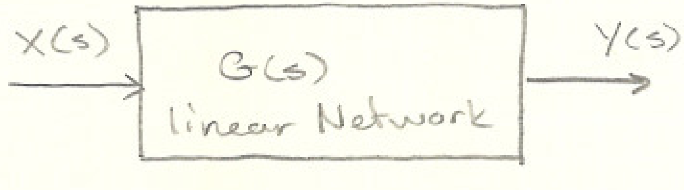


If $V_1(s) = 1$

then $V_2(t) = \frac{1}{2\omega} t \sin \omega t$

STEADY STATE RESPONCE OF NETWORKS.

in general, a linear network can be represented by:



$$T.F. = G(s) = \frac{Y(s)}{X(s)}$$

if s is replaced by jw, a steady state responce is obtained, assuming zero initial conditions, and since the input and output can either be voltages or currents, then we have 4 possible T.F.

$$\frac{V_2}{V_1} \qquad \frac{I_2}{V_1} \qquad \frac{I_2}{I_1} \qquad \frac{V_2}{I_1}$$

Transfer impedance

current gain

Transfer admitance

voltage gain

Since $G(j\omega)$ is a complex quantity, then we may write

$$G(j\omega) = |G(j\omega)| \angle B$$

which is a phasor, and since $G(j\omega)$ is a complex quantity in general, then

$$G(j\omega) = \frac{N(j\omega)}{D(j\omega)}$$

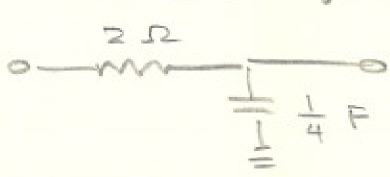
- when $N(j\omega)$ is equated to zero will give the zeros of the T.F.

- when $D(j\omega)$ is equated to zero, it will give use the poles of the T.F.

Ex:

$$G(s) = \frac{2}{s+2}$$

which comes from the following circuit.



when the initial conditions are zero.

let $s = j\omega$, then the steady state TF. is given by:

$$G(j\omega) = \frac{2}{j\omega + 2} = \frac{2}{2 + j\omega}$$

we want to change this into

$$|G(j\omega)| \angle \beta$$

where

$|G(j\omega)| \rightarrow$ amplitude.

$\beta(\omega) \rightarrow$ phase angle.

therefore

$$|G(j\omega)| = \left| \frac{2}{2 + j\omega} \right| = \frac{2}{\sqrt{2^2 + \omega^2}}$$

$$\beta = -\tan^{-1}\left(\frac{\omega}{2}\right)$$

If $v_1(t) = 5 \sin 2t$, then $v_1(t)$ can be written as a phasor

$$V_1(j\omega) = 5 \angle 0^\circ \quad \text{where} \quad \omega = 2$$

then.

$$G(j2) = \frac{2}{2 + j2} = \left| \frac{2}{2 + j2} \right| \angle \beta$$

$$= \frac{1}{\sqrt{2}} \angle -45^\circ$$

$\therefore V_2(j\omega) = G(j\omega)\, V_1(j\omega)$

$$= \left[\frac{1}{\sqrt{2}} \angle 45^\circ\right] 5$$

and the steady state responce is phasor.

$$V_2(j\omega) = \frac{5}{\sqrt{2}} \angle 45^\circ$$

$$V_2(t) = \frac{5}{\sqrt{2}} \sin(2t - 45^\circ)$$

for linear circuits, the transform of interest are the ratios of polynomicls in s, such functions are called rational functions

To find the inverse of laplace transform, we must find the waveforms corrisponding to the rational function of the form.

$$F(s) = k \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

k is a scaller

z_i are zeros

p_i are poles

if $n > m$, there are more finite poles and then the infinite zeros and $F(s)$ is called a proper rational function

If the denominator of $F(s)$ has no repeated roots, then $F(s)$ is said to have simple poles

If proper rational function has only simple poles, then it can be decomposed into

$$F(s) = \frac{k_1}{(s-P_1)} + \frac{k_2}{(s-P_2)} + \cdots + \frac{k_n}{(s-P_n)}$$

$$\mathcal{L}^{-1}(F(s)) = k_1 e^{-P_1 t} + k_2 e^{-P_2 t} + \cdots + k_3 e^{-P_n t}.$$

$k_1, k_2, \ldots, k_n$ are known as residues.

recall the cover up method.

$$k_i = \Big[(s-P_i)\, F(s)\Big]_{s=P_i}$$

REGAURDING COMPLEX POLES

we may extend the above for complex roots, consider the following:

$$F(s) = \underbrace{\cdots\cdot}_{\text{real poles}} + \frac{\bar{k}}{(s+\alpha-j\beta)} + \frac{\bar{k}^*}{(s+\alpha-j\beta)}$$

The residues $\bar{k}$ & $\bar{k}^*$ at the conjugate poles are themselves conjugates b/c $F(s)$ is $\mathbb{R}$

then $\bar{k} = |\bar{k}|e^{j\theta}$ & $\bar{k}^* = |\bar{k}|e^{j\theta}$

then the waveforms corresponding to the above 2 terms are ...

$$f(t) = \underbrace{\cdots\cdots}_{\text{real part}} + |\bar{k}|e^{j\theta}e^{(-\alpha+j\beta)t} + |\bar{k}|e^{-j\theta}e^{(-\alpha-j\beta)t}$$

rearranging we get.

$$f(t) = \ldots + 2|\bar{k}|e^{-\alpha t}\left[\frac{e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}}{2}\right]$$

or

$$f(t) = \ldots + 2|\bar{k}|e^{-\alpha t}\cos(\beta t+\theta)$$

for h/w try to find $\mathcal{L}^{-1}$

$$F(s) = \frac{20(s+3)}{(s+1)(s^2+2s+5)}$$

COMPLEX NUMBERS

complex numbers were invented to permit the extraction of roots of negative numbers.

NOTATION

$$j = \sqrt{-1}$$

$$\bar{z} = a + jb \quad \text{(rectangular form)}$$

$$\bar{z} = |\bar{z}|\, e^{j\theta} = |\bar{z}| \angle \theta \quad \text{(polar form)}$$

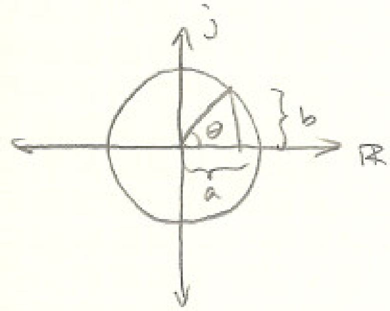
$$|\bar{z}| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

we may write

$$|\bar{z}| \cos\theta = a$$

$$|\bar{z}| \sin\theta = b$$



or

$$|\bar{z}|\, e^{j\theta} = |\bar{z}| (\cos\theta + j\sin\theta)$$

Ex: $\bar{z} = 3 + j4 = 5e^{j53.1°} = 5 \angle 53.1°$

Ex: $\bar{z} = -4 + j2 = 4.472e^{j173.4°} = 4.472 \angle 173.4°$

DIVISION.

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{|\bar{z}_1|}{|\bar{z}_2|} e^{j(\theta_1 - \theta_2)}$$

ADDITION

$$\bar{z}_1 + \bar{z}_2 = a_1 + jb_1 + a_2 + jb_2 = (a_1 + a_2) + j(b_1 + b_2)$$

MULTIPLYCATION OF CONJEGATES

$$\bar{z}_1 \cdot \bar{z}^* = a^2 + b^2 = |\bar{z}|^2$$

Ex:

$$F(s) = \frac{20(s+3)}{(s+1)(s^2+2s+5)}$$

SOL: F(s) has a simple pole at $s=-1$ and a pair of complex conjugate poles which are given by

$$(s^2+2s+5) = (s+1-j2)(s+1+j2)$$

partial fractions of F(s)

$$F(s) = \frac{k_1}{s+1} + \frac{\bar{k}_2}{s+1-j2} + \frac{\bar{k}_2^*}{s+1+j2}$$

using coverup algorithm, then.

$$k_1 = \frac{20(s+3)}{s^2+2s+5}\bigg|_{s=-1} = 10$$

$$\bar{k}_2 = \frac{20(s+3)}{(s+1)(s+1+j2)}\bigg|_{s=-1+j2} = -5-j5 = 5\sqrt{2}\, e^{j\frac{5}{4}\pi}$$

$$\overline{k_2}^* = -5 + j5 = 5\sqrt{2}\, e^{-j\frac{5}{4}\pi}$$

$$F(s) = \frac{10}{s+1} + \frac{5\sqrt{2}\, e^{j\frac{5}{4}\pi}}{s+1-j2} + \frac{5\sqrt{2}\, e^{-j\frac{5}{4}\pi}}{s+1+j2}$$

$$f(t) = \left\{10e^{t} + 10\sqrt{2}\, e^{-t} \cos\left(2t + \frac{5}{4}\pi\right)\right\} u(t)$$

<u>note:</u>

$$\frac{\overline{k}}{s+\alpha-j\beta} + \frac{\overline{k}^*}{s+\alpha+j\beta} = F(s)$$

$$\mathcal{L}^{-1}\left[F(s)\right] = 2|\overline{k}|e^{-\alpha t}\cos(\beta t + \theta)$$

$$\frac{\overline{k}}{(s+\alpha-j\beta)^2} + \frac{\overline{k}^*}{(s+\alpha+j\beta)^2} = F(s)$$

$$\mathcal{L}^{-1}\left[F(s)\right] = 2t|\overline{k}|e^{-\alpha t}\cos(\beta t + \theta)$$

where $\overline{k} = |\overline{k}|e^{j\theta}$

$\overline{k}^* = |\overline{k}|e^{-j\theta}$

MULTIPLE POLES

consider

$$F(s) = \frac{k(s - z_1)}{(s - p_1)(s - p_2)^2}$$

it has a multiple pole a $s = p_1$ and $s = p_2$

$$F(s) = \frac{1}{s - p_2}\left(\frac{k(s - z_1)}{(s - p_1)(s - p_2)}\right)$$

and

$$F(s) = \frac{1}{s - p_2}\left[\frac{c_1}{s - p_1} + \frac{k_{22}}{s - p_2}\right]$$

then

$$F(s) = \frac{c_1}{(s - p_1)(s - p_2)} + \frac{k_{22}}{(s - p_2)^2}$$

Expanding.

$$\frac{c_1}{(s - p_1)(s - p_2)} = \frac{k_1}{(s - p_1)} + \frac{k_{21}}{(s - p_2)}$$

$\therefore$

$$F(s) = \frac{k_1}{s - p_1} + \frac{k_{21}}{s - p_2} + \frac{k_{22}}{(s - p_2)^2}$$

$$\mathcal{L}^{-1}[F(s)] = \left\{k_1 e^{p_1 t} + k_{21} e^{p_2 t} + k_{22}\, t\, e^{p_2 t}\right\} u(t)$$

EX: find the inverse.

$$F(s) = \frac{4(s+3)}{s(s+2)^2}$$

$$F(s) = \frac{1}{(s+2)}\underbrace{\left[\frac{4(s+3)}{s(s+2)}\right]}_{F_0(s)}$$

$$F_0(s) = \frac{4(s+3)}{s(s+2)} = \frac{A_1}{s} + \frac{A_2}{s+2}$$

coverup technique

$$A_1 = \frac{4s(s+3)}{s(s+2)}\Bigg|_{s=0} = 6$$

$$A_2 = \frac{\cancel{(s+2)}\,4(s+3)}{s\cancel{(s+2)}}\Bigg|_{s=-2} = -2$$

$\therefore$

$$F(s) = \frac{1}{s+2}\left[\frac{6}{s} - \frac{2}{s+2}\right]$$

$$F(s) = \underbrace{\frac{6}{s(s+2)}}_{\text{expand.}} - \frac{2}{(s+2)^2}$$

$$F(s) = \frac{3}{s} - \frac{3}{s+2} - \frac{2}{(s+2)^2}$$

$$\mathcal{L}^{-1}[F(s)] = \{3 - 3e^{-2t} - 2te^{-2t}\}\,u(t)$$