LIMITS INVOLVING INFINITY.

- Infinite limits.

$f(x) = \frac{1}{x}$ $\quad x \to 0^-$, $x \to 0^+$

$\lim_{x \to 0^-} f(x) = -\infty$ (but by definitition the limit doesn't exist)

$f(x) = \frac{1}{x^2}$

$\lim_{x \to 0} f(x) = \infty$

- Limits at infinity.

$\lim_{x \to \infty} f(x) = L$ means that when x becomes very large, then f(x) approaches L.

EX. $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = ?$

Solution.

Start by dividing everything by $\left(\frac{1/x^2}{1/x^2}\right)$

$$\lim_{x \to \infty} \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2} = \frac{3}{5}$$

EX. $\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right)$

Solution

Start looking for a way to eliminate x.

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}\right)$$

$$\lim_{x \to \infty} \left(\frac{x^2 + 1 - x^2}{\sqrt{x^2+1} + x}\right) = 0$$

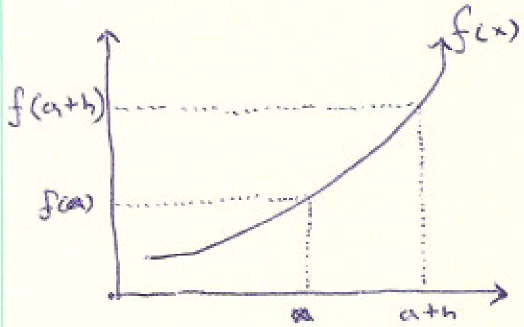
§2.7 DERIVATIVES.

The derivative of a function f at a, denoted by $f'(a)$, is defined as,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the limit exists.

GRAPHICALLY:



$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

- APPLICATIONS OF DERIVATIVES.

BACK IN §2.1 & §2.6 WE TALKED ABOUT TANGENTS AND RATES OF CHANGE, DERIVATIVES IS ABOUT FIND TANGENTS AND RATES OF CHANGE.

EX. Velocity problem.

given $s = f(t)$, the position of a particle at time t.

t is in sec.
s is in m.

find average velocity the $\Delta x \to 0$ to find $s'(t)$

§ 2.8 THE DERIVATIVE OF A FUNCTION.

- $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$. this is the derivative at a given point a.

if a is replaced by x then

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$. this is the derivative of the entire function.

- HIGHER ORDERS.

$f''(x)$ SECOND DERIVATIVE (the derivative of the derivative).

$f^n(x)$ n^{th} DERIVATIVE.

- OTHER NOTATIONS.

$$f'(x) = y' = \frac{df(x)}{dx} = \frac{dy}{dx} = \frac{d}{dx} f(x) = Df(x)$$

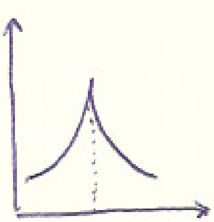
$$f''(x) = y'' = \frac{d^2y}{dx^2} = \frac{d^2}{dx^2} f(x). = D^2 f(x).$$

- DIFFERENTIABLE FUNCTIONS.

A FUNCTION IS DIFFERENTIABLE AT a IF $f'(a)$ EXISTS.

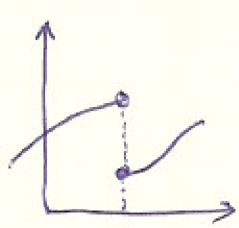
A FUNCTION IS DIFFERENTIABLE AT AN INTERVAL IF IT IS DIFFERENTIABLE AT EVERY NUMBER IN THE INTERVAL.

A FUNCTION FAILS TO BE DERIVED AT a



a

CORNER



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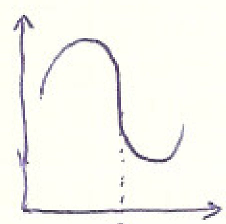
DISCONTINUITY.



a

VERTICAL TANGENT.

CHAPTER 2 REVIEW.

§ 2.1 Definition of a limit.

$$\lim_{x \to a} f(x) \neq f(a) \quad \text{CAN HAPPEN.}$$

§ 2.3
- ABLE TO DO DIRECT SUBSTUTION.
- RATIONAL LIMITS.
- PIECEWISE LIMITS.

§ 2.4
- CONTINUITY

§ 2.5
- LIMIT APPROACHING INFINITY.
-

§ 2.7
- DEFINITION OF A DERIVATIVE.

§ 2.8
- DERIVATIVE CAN BE A FUNCTION.
- HIGHER ORDER DERIVATIVES.
- UNDIFFERENTIABLE FUNCTIONS.

CHAPTER 3.

§ 3.1 DERIVATIVES OF POLYNOMIALS AND EXP. FUNCTIONS.

- POLYNOMIALS. AND POWER FUNCTIONS.

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$
$$f(x) = cx^n$$
$$f'(x) = cnx^{n-1}$$
$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

EX.
$$f(x) = x^{10} + 2x^{20}$$
$$f'(x) = 10x^9 + 40x^{19}$$

EX.
$$f(x) = \frac{1}{x^3} = x^{-3} \qquad f(x) = \sqrt[3]{x^2} = x^{2/3}$$
$$f'(x) = -3x^{-4} \qquad f'(x) = \frac{2}{3}x^{-1/3}$$

- DERIVATIVE OF NATURAL EXPONENIAL FUNCTION.

$$\frac{d}{dx}\left[e^x\right] = e^x$$

- DERIVATIVES OF EXPONENTAL FUNCTIONS.

$$\frac{d}{dx}\left[a^x\right] = \ln a\left(a^x\right)$$

§ 3.2 THE PRODUCT AND QUOENT RULES.

- if g(x) and h(x) are both differentiable, then,

$$\frac{d}{dx}\left[g(x)\cdot h(x)\right] = g'(x)\cdot h(x) + g(x)\cdot h'(x).$$

- if g(x) and h(x) are both differentiable, then,

$$\frac{d}{dx}\left[\frac{g(x)}{h(x)}\right] = \frac{g'(x)\cdot h(x) - g(x)\cdot h'(x)}{[h(x)]^2}$$

§ 3.4 DERIVATIVES OF TRIG FUNCTIONS.

- $\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$
- $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x).$
- $\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$
- $\frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$
- $\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\cdot\tan(x).$
- $\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cdot\cot(x).$

§ 3.5 CHAIN RULE.

- $f(g(x)) = f'(g(x))\cdot g'(x).$

or

$f(g(x))$ let $u = g(x)$. $\therefore$ $f(u)$.

$$\frac{df}{du}\cdot\frac{du}{dx} = f'(u)$$

§ 3.6 IMPLICIT DIFFERENTIATION.

- EXPLICIT vs. IMPLICIT.

EXPLICIT.

$$y = f(x).$$

IMPLICIT.

$$F(x, y, C) = 0$$

eg.

$$(x-h)^2 + (y-k)^2 - r^2 = 0$$

EX.

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[25\right]$$

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

- DERIVATIVES OF INVERSE TRIG FUNCTIONS.

$$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$$

§ 3.7 DERIVATIVES OF LOG. FUNCTIONS.

- $$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}$$

PROOF:

$$y = \log_a x$$

$$x = a^y$$

implicitly differentiate.

$$1 = a^y (\ln a)\, y'$$

$$y' = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

EXAMPLE,

$$y = x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} y' = \ln x + \frac{x}{x}$$

$$y' = y(\ln x + 1)$$

$$y' = x^x(\ln x + 1)$$

EX.

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{1}{y}y' = \frac{3}{4x} + \frac{2x}{2(x^2+1)} - \frac{15}{3x+2}$$

$$y' = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

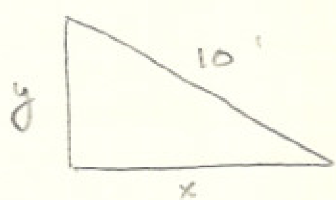
CHAPTER 3 REVIEW

- 3.1, 3.2, 3.4, 3.5, 3.6, 3.7 → must understand.

CHAPTER 4. APLICATIONS OF DIFFERENTIATION.

§ 4.1 RATES OF CHANGE.

EX



$x = 6'$ $\therefore y = 8'$

$\frac{dy}{dt} = -1\ ft/s$

$$x^2 + y^2 = 10^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(6)\frac{dx}{dt} + 2(8)(-1) = 0$$

$$\frac{dx}{dt} = 4/3\ ft/s.$$

§4.1 RATES OF CHANGE.

- time rate of change.
- tangent to a curve at a point.

point slope at point $(x_1, f(x_1))$

$$(y - f(x)) = f'(x_1)(x - x_1) \qquad \text{tangent.}$$

$$(y - f(x)) = \frac{-1}{f'(x_1)}(x - x_1) \qquad \text{normal.}$$

§4.2 MAX AND MIN VALUES.

- TERMS.

  - The Absolute (global) max./min values.

    $f(c) \geq f(x)$ for all the Domain of $f(x)$ this would be the absolute max.

    $f(c) \leq f(x)$ for all the Domain of $f(x)$ this would be the absolute min.

  - The locate (Reclitive) min/max values.

    $f(c) \geq f(x)$ for all values near to c, this would be a local max.

    $f(c) \leq f(x)$ for all values near to c, this would be a local min.

- THEOREMS.

Extreme value theorom, has 2 requirements, there must be continuity, and a closed interval. Then ther must be a min, max.

Fermat therom. is the nessasary condition of local min/max

$f'(c) = 0.$ but not in all cases.

- CRITICAL NUMBER

critical number of f is a number c where, in the domain of f.

$$f'(c) = 0$$
$$f'(c) = \text{doesn't exist.}$$

EX.

$$f(x) = x^{4/5}(x-4)^2 \qquad x \in [-1, 1]$$

$$f'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + x^{4/5}(2)(x-4)$$
$$= 2x^{-1/5}(x-4)\left[\frac{2}{5}(x-4) + x\right]$$
$$= \frac{2(x-4)(7x-8)}{5\sqrt[5]{x}}$$

Critical pts,

$x = 0$, $f'(0)$ doesn't exist.

$x = 4$, $x = \frac{8}{7}$, $f'(0) = 0$

$\therefore$ there is only one critical number within $(-1, 1)$

then evaluate the end points....

$f(-1) = 25$

$f(1) = 9$

$\therefore$ global max at $x = -1$

global min at $x = 1$

§ 4.3 DERIVATIVES AND SLOPES OF CURVES.

- CURVE SCETCHING.
  - $f'(x)$
    - $f'(x)=0$, local min/max, transition pt, etc.
    - $f'(x)>0$, function will be i/c over interval.
    - $f'(x)<0$, function will be d/c over interval.
  - $f''(x)$.
    - $f''(x)=0$, inflection point, charge concavity.
    - $f''(x)>0$, function is concave upward.
    - $f''(x)<0$, function is concave downward.
  - FIRST DERIVATIVE TEST.
    - $f'(x)$ changes from pos. to neg. at point c, then pt. c is a local max.
    - $f'(x)$ changes from neg. to pos. at a point c, then pt. c is a local min.
    - $f'(x)$ does not change signs then there is no local min/max.
  - SECOND DERIVATIVE TEST.
    - $f'(c)=0$, $f''(c)>0$, then local min
    - $f'(c)=0$, $f''(c)<0$, then local max.
    - $f''(c)=0$, then inflection pt. at c.

EX.

SKETCH THE CURVE $f(x) = x^4 - 4x^3$

roots at $x = 0, 4$.

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

critical pts. $x = 0, 3$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

inflection pts. $x = 0, 2$

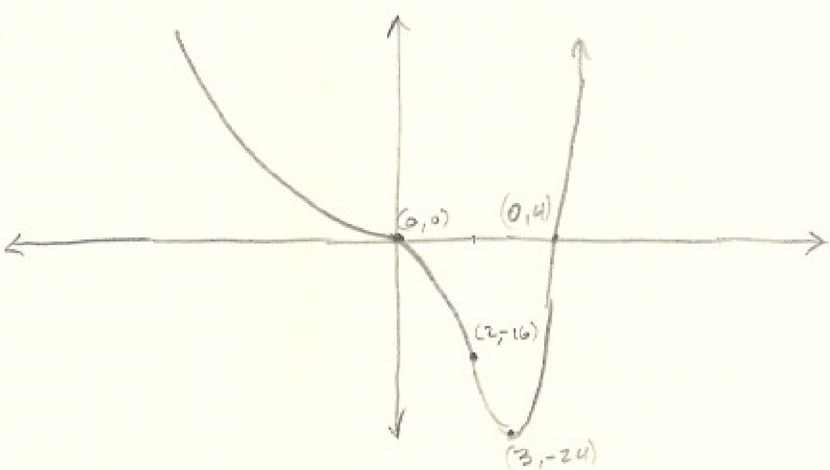
$f''(3) > 0$, therefore $x = 3$ is minimum.

$f(0) = 0$

$f(2) = -16$

$f(3) = -27$.

$\therefore$



Should also check pts of interest intervals.

	$f'(x)$	$f''(x)$.
$-\infty, 0$	$-$	$+$
$0, 2$	$-$	$-$
$2, 3$	$-$	$+$
$3, 4$	$+$	$+$
$4, \infty$	$+$	$+$

§ 4.5 INDETERMINATE FORMS (OF LIMITS) AND L'HOSPITAL'S RULE.

- $\lim_{x \to a}$ $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$, ∞^{∞}, 0^{∞}

  $\lim_{x \to \pm\infty}$

these are all forms of indeterminate limits.

- L'HOSPITAL's RULE

  this allows us to deal w some cases.

  iff $f(x)$ and $g(x)$ are differentiable, and $g'(x) \neq 0$ near a.

  suppose

$$\lim_{x \to a} f(x) = 0 \qquad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \qquad \lim_{x \to a} g(x) = \pm\infty$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

if the limit exists or is equal to $\pm\infty$

EX.

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \frac{0}{0}$$

$$\therefore \lim_{x \to 1} = \frac{1/x}{1} = 1$$

EX. form $0 \cdot \infty$

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0} \frac{\ln x}{1/x} = \frac{\infty}{\infty}$$

$$\lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} \left(\frac{1}{x} \cdot \frac{-x^2}{1} \right)$$

$$\lim_{x \to 0^+} x = 0.$$

EX. form $\infty - \infty$

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$$

$$\lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$\lim_{x \to (\pi/2)^-} \left(\frac{1 - \sin x}{\cos x} \right) = \frac{0}{0}$$ now use l'hostipal's

$$\lim_{x \to (\pi/2)^-} \left(\frac{-\cos x}{\sin x} \right) = 0.$$

EX forms $0^0, \infty^\infty, 0^\infty, \infty^0, 1^\infty$

$$\lim_{x \to a} f(x)^{g(x)}$$

$$\lim_{x \to a} g(x) \ln f(x).$$ this finds limit of $\ln y$.

$$\lim_{x \to a} \frac{\ln f(x)}{1/g(x)}$$

$$\lim_{x \to a} \left(\frac{f'(x)}{f(x)} \cdot \frac{d}{dx}\left(\frac{1}{g(x)}\right) \right) = c = \lim_{x \to a} \ln y$$

$$\lim_{x \to a} y = e^c$$

CHAPTER 4 REVIEW

- RATES OF CHANGE
- EXTREME VALUES AND OPTIMIZATION.
  - global max/min vs. local max/min.
  - critical numbers.
  - extreme values in an interval.
  - first derivative test.
- f' and f'' and the slope of the curve.
- L'HOSPITAL'S RULE.

CHAPTER 5 INTEGRALS.

§ 2.10 & § 4.9. ANTIDERIVATIVES.

- Definition.

  A function F is called an antiderivative of f on an interval if $F'(x)=f(x)$ for all x in the integral.

- Family of anti-derivatives.

  if F is the anti-derivative of f, then $F+C$ is also the anti derivative of f. where C is an arbitrary constant.

- Notation. (indefinite integral)

  $\int f(x)\,dx = F(x)+c$

  (RHS is a function)

EX.

$f(x) = x^n$ $\qquad$ $\int f(x) = \frac{x^{n+1}}{n+1}$ $\quad x \neq -1.$

$f(x) = \frac{1}{x}$ $\qquad$ $\int f(x) = \ln x$

$f(x) = \sin x$ $\qquad$ $\int f(x) = -\cos x$

EX.

$$f'(x) = e^x + \frac{20}{1+x^2}$$

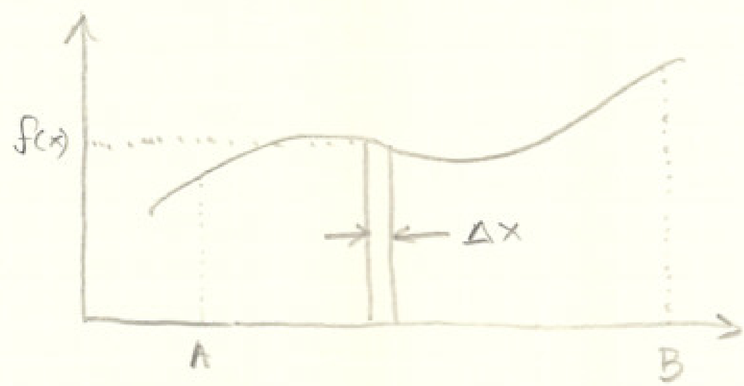
$$\int f'(x)\,dx = e^x + 20\tan^{-1}x + c$$

$$-2 = e^0 + 20\tan^{-1}(0) + c$$

$$c = -3$$

$$\therefore\ f(x) = e^x + 20\tan^{-1}x - 3$$

§ 5.1 AREAS AND DISTANCES.



we take an unlimited # of divisions.
then add them all up.

$$A = \sum_{i=1}^{n} f(x)\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x)\Delta x$$

if the lim exist we write it as.
a definate integral.

$$A = \int_A^B f(x)\,dx.$$

§ 5.2

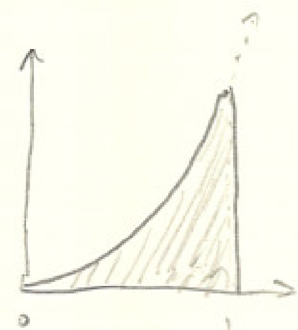
- DEFINITION.

  If f is a continuous function defined on $[a,b]$ Devide $[a,b]$ into n subintervals of equal width, $\Delta x = (b-a)/n$, let $x_0 = a, x_1, x_2 \ldots x_n = b$ be the endpoints of the intervals. Choose the point $x_1, x_2 \ldots$ so that x_i lies in the i^{th} subinterval $[x_{i-1}, x_i]$, then the definite integral of f from a to b.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

EX.

$y = x^2$



$$\int_0^1 f(x)\,dx = \int_0^1 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1}{3} + C\right] - \left[\frac{0}{3} + C\right]$$

$$= \frac{1}{3}$$

§ 5.4 THE FUNDIMENTAL THEROM OF CALCULUS.

Suppose f is continuous on $[a,b]$, then

- the function defined by,

$$g(x) = \int_a^x f(x)\,dx \qquad a \le x \le b.$$

- $\int_A^B f(x)\,dx = F(b) - F(a)$, where F is an antiderivative of f, that is $F'(x) = f(x)$.

§ 5.3 EVALUATING DEFINITE INTEGRALS.

$$\int_A^B f(x)\,dx = F(b) - F(a).$$

§ 5.5 INDEFINITE / DEFINITE INTEGRALS.

boils down to just finding the antiderivative.

SUBSTITUTION RULE.

similar to reverse chain rule.

EX.

$$\int x^3 \cos(x^4 + 2)\,dx$$

let $u = x^4 + 2$

$du = 4x^3\,dx$

$\frac{du}{4} = x^3\,dx$

$\therefore \frac{1}{4}\int \cos u\,du$

$\frac{\sin(x^4+2)}{4} + C$

EX.

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$$

§ 5.5 SUBSTITUTION RULE

EX.

$$\int \cos^3 x \, dx$$

$$\int \cos^2 x \cos x \, dx$$

$$\int (1 - \sin^2 x) \cos x \, dx$$

let $u = \sin x$
$du = \cos x \, dx$

$$\int (1 - u^2) \, du$$

$$u - \frac{u^3}{3} + C$$

$$\sin x - \frac{\sin^3 x}{3} + C$$

SUBSTITUTION RULE (FOR DEFINIT INTEGRALS).

$$\int_a^b f(g(x)) \cdot g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

EX.

$$\int_0^4 \sqrt{2x+1} \, dx$$

let $u = 2x + 1$
$du = 2dx$

$dx = \frac{du}{2}$

$$\frac{1}{2}\int_1^9 u^{1/2} du$$

$$\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^9$$

$$\frac{1}{3}\left[27 - 1\right]$$

$$\frac{26}{3}$$