$$\lim_{n\to\infty}\left(\frac{n+1}{3n}|x+2|\right) = \frac{|x+2|}{3}$$

since $\frac{|x+2|}{3} < 1 \quad \therefore \quad x+2 < 3$

$\therefore R = 3$

$x = a - R = -2 - 3 = -5$

series becomes

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum(-1)^n \cdot n$$

$\therefore$ not alterating series. $\therefore -5$ is divergent.

$x = a + R = 1$

series becomes.

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum n$$

$\therefore$ divergent at 1 $\therefore$ interval is $(-5, +1)$

§ 8.7 TALOR AND MACLAURIN SERIES.

POWER SERIES. ① CONVERGENCE

② R

③ [ ) ( ] ( ) [ ]

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

given f.

the question is how to determine the C_n's in terms of given function f.

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \ldots$$

1. $x = a \quad f(a) = C_0 \quad \therefore \quad C_0 = f(a)$

2. $f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2$

$x = a \quad f'(a) = C_1 \quad \therefore \quad C_1 = f'(a)$

3. $f''(x) = 2C_2 + 3 \cdot 2C_3(x-a) + \ldots$

$x = a \quad f''(a) = 2C_2 \quad \therefore \quad C_2 = \frac{f''(a)}{2}$

4. $f'''(x) = 3 \cdot 2C_3(x-a) + \ldots$

$x = a \quad f'''(a) = 6C_3 \quad \therefore \quad C_3 = \frac{1}{6} f'''(a).$

$$\therefore \quad C_n = \frac{1}{n!} f^n(a)$$

$$\Sigma C_n (x-a)^n = \underbrace{\Sigma \frac{1}{n!} f^n(a)(x-a)^n}_{\text{special power series known as taylor series.}}$$

it is also known as the taylor series of $f(x)$ where it is understood that $f^0 = f$, $0! = 1$

if $a = 0$ the taylor series becomes

$$\underbrace{\Sigma \frac{1}{n!} f^{(n)}(0) x^n}_{\text{MACLAURIN SERIES.}}$$

EX.

find the MacLauren Series $f(x) = e^x$

SOLUTION $a = 0$

$$C_n = \frac{1}{n!} f^{(n)}(0) = \frac{1}{n!} e^0 = \frac{1}{n!}$$

$$\therefore \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

set $x = 1$

$$e = 2 + \frac{1}{2} + \frac{1}{6}$$

EX.

find the taylor Series. for $f(x) = e^x$, centered at $a = 2$

SOLUTION: $a = 2$

$$C_n = \frac{1}{n!} f^{(n)}(a) = \frac{1}{n!} e^2$$

$$\therefore \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} e^2 (x-2)^n$$

EX

find the maclaurin Series for $\cos x$.

SOLUTION:

$$\begin{aligned} f(x) &= \cos(x) &= 1 \\ f'(x) &= -\sin(x) &= 0 \\ f''(x) &= -\cos(x) &= -1 \\ f'''(x) &= \sin(x) &= 0 \end{aligned}$$

$$\therefore \quad C_0 = \frac{f(0)}{0!} = 1$$

$$c_1 = \frac{f'(0)}{1!} = 0$$

$$c_2 = \frac{f''(0)}{2!} = \frac{-1}{2!}$$

$$c_3 = \frac{f'''(0)}{3!} = 0$$

$$c_4 = \frac{f^4(0)}{4!} = \frac{+1}{4!}$$

$$\therefore \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^2 - \frac{1}{6!}x^2$$

$$= \Sigma (-1)^n \frac{x^{2n}}{2n!}$$

COMMONLY USED / IMPORTANT SERIES. p 612

§8.6, §8.7, §8.9 APPLICATIONS.

mathimatical applications

obtain new series through algerbra. calculus.

evaluate limits

evaluate integrals

practical taylor polynomials.

EX.

find Taylor Series for MacLaurin.

1) e^{x+1}

2) $\sin x / x \quad x \neq 0$

then find $\lim_{x \to 0} \frac{\sin x}{x}$ and $\int \frac{\sin x}{x} dx$

SOLUTION:

1) $e^{x+1} = e\, e^{x}$

$$\therefore \quad e^{x+1} = e \sum \frac{x^n}{n!}$$

2) $$\frac{\sin x}{x} = \frac{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{x}$$

$$= \sum \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\lim_{x \to 0} \sum \frac{(-1)^n x^{2n}}{(2n+1)!} = \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) = 1$$

$$\int \lim_{x \to 0} \sum \frac{(-1)^n x^{2n}}{(2n+1)!} = \int \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right)$$

$$= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \dots$$

$$= \sum \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} + C$$

EX. $\int e^{-x^2} dx$ can't be integrated through essiential functions.

SOLUTION

$$e^x = \sum \frac{x^n}{n!}$$

$$\therefore \quad e^{-x^2} = \sum \frac{(-x^2)^n}{n!} = \sum (-1)^n \frac{x^{2n}}{n!}$$

$$\therefore \int e^{-x^2} dx = \int \sum (-1)^n \frac{x^{2n}}{n!}\, dx$$

$$= \sum \frac{(-1)^n}{n!} \int x^{2n}\, dx$$

$$= \sum \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} + C$$

OTHER COMMONALY USED SERIES.

- BINOMIAL
- TRIGONOMIC SERIES.

TAYLOR POLYNOMIAL

taylor polynomial to the nth degree.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

comments

- $T_n(x) \neq f(x)$

  $T_n(x)$ is an aproximation of $f(x)$
- $f(x) = \lim_{n \to \infty} T_n(x)$
- $R(x) = f(x) - T_n(x)$ remainder.

  $$= \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

how many terms to use?

$V_{accual} = f(x)$

$V_{approx} = T_n(x)$ is rounded of to i decimal places.

$$|V_{acual} - V_{approx}| < 5 \times 10^{-(i+1)}$$

CHAPTER 8 REVIEW.

- Convergence of a sequence.

  $\lim_{n\to\infty} a_n$ if exists - convergnt
  doesn't exist - divergent.

- Convergence of a series.
  - partial sum. $\{S_n\}$

  $\lim_{n\to\infty} S_n$ exists then convergent.

- Convergence test
  - All tests.
- MacLaurin Series
  - Special case of taylor series. when $a=0$
- Taylor Series.
  - Power Series when a is any number.
- Applications.
  - find a knew series.
  - differentiation, limit, integration.

HYPERBOLIC & INVERVE HYPERBOLIC FUNCTIONS.

- Some special combinations of e^x, e^{-x}
- Solution to ODE.
- Basic definition

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x - e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\coth(x) = \frac{1}{\tanh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

- Identies.

trigonomic $\Rightarrow$ negative angles
hyperbolic $\Rightarrow$ negative argument.

$$\sin(-x) = -\sin(x)$$

$$\sinh(-x) = -\sinh(x)$$

Addition/Subtraction

$$\sinh(x \pm y) = \sin x \cos y + \cos x \sin y.$$

Double angle.

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

Half angle

$$\sin\left(\frac{x}{n}\right) \qquad \sinh\left(\frac{x}{2}\right)$$

Sum/difference/Product.

$\sin x \pm \sin y$ $\qquad$ $\sinh x \pm \sinh y$.

$\sin x \cdot \sin y$.

- Derivatives.

$$\frac{d}{dx}\left[\sinh x\right] = \frac{d}{dx}\left[\frac{e^{x} - e^{-x}}{2}\right] = \cosh x$$

$$\frac{d}{dx}\left[\cosh x\right] = \sinh x.$$

EX. $$\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2}$$

$$\frac{3 + 1/3}{2} = \frac{2}{3}$$

EX. $$\tanh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}}$$

$$= \frac{x - 1/x}{x + 1/x} \cdot \frac{x}{x} = \frac{x^2 - 1}{x^2 + 1}$$

EX. $$\sinh(x) = \frac{3}{4}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\therefore \cosh^2(x) = 1 + \left(\frac{3}{4}\right)^2 = \frac{25}{16}$$

$$\therefore \cosh^2(x) = \frac{5}{4}$$

EX. $$\tanh(3x) = f(x)$$

$$f'(x) = \text{sech}^2(3x) \cdot 3$$

EX. $\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1$

INVERSE HYPERHOLIC FUNCTIONS.

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right) \qquad x \in \mathbb{R}$$

$$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right) \qquad x \geq 1$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad |x|<1$$

$$\frac{d}{dx}\left[\sinh^{-1}(x)\right] = \frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+1}\right)\right]$$

$$= \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left[\cosh^{-1}(x)\right] = \pm\frac{1}{\sqrt{x^2-1}}$$

EX.

$$f(x) = x \cdot \ln\left(\operatorname{sech}^{-1}(4x)\right)$$

$$f'(x) = \ln\left(\operatorname{sech}^{-1}(4x)\right) + x \cdot \frac{4}{\operatorname{sech}^{-1}(4x)}\left(-\operatorname{sech} 4x \right.$$

$$\left. \tanh 4x\right)$$

EX.

$$\int \sinh(2x)\,dx = \frac{1}{2}\cosh(2x) + c$$

COMPLEX NUMBERS.

$a + bi$ — a, b: real numbers.
$i = \sqrt{-1}$, imageneary unit.