- PROPERTIES OF DEFINITE INTEGRALS.
  - $\int_A^B f(x)\,dx$

    this is accually a number. It does not depend on x. X is called a dumby variable and can be replaced $\overline{w}$ some other variable.

  - $\int_a^b c\,dx = c(b-a)$

    $\int_a^b (g(x) \pm f(x))\,dx = \int_a^b g(x)\,dx \pm \int_a^b f(x)\,dx$

    $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

    $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

    $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

    $\int_a^a f(x)\,dx = 0$

  - iff $f$ is an even function.

    $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

    iff $f$ is odd

    $\int_{-a}^a f(x)\,dx = 0$

EX.

$$f(x) = x^{2/3}(6-x)^{1/3}$$

$$= \frac{2}{3}x^{-1/3}(6-x)^{1/3} + x^{2/3}\left(\tfrac{1}{3}\right)(6-x)^{-2/3}(-1)$$

$$= \frac{1}{3}x^{2/3}(6-x)^{1/3}\left(2x^{-1} - (6-x)^{-1}\right)$$

$$= \frac{1}{3}x^{2/3}(6-x)^{1/3}\left(\frac{2}{x} - \frac{1}{6-x}\right)$$

$$= \frac{1}{3}x^{2/3}(6-x)^{1/3}\left(\frac{2(6-x) - x}{x(6-x)}\right)$$

$$= \frac{1}{3}x^{2/3}(6-x)^{1/3}\left(\frac{12-3x}{x(6-x)}\right)$$

## § 5.6 INTEGRATION BY PARTS.

- $\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'(x)\,g(x) + f(x)\,g'(x).$

$$\int \frac{d}{dx}\left[f(x)\cdot g(x)\right]dx = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

$$f(x)\cdot g(x) = \int f'(x)\cdot g(x)\,dx + \int f(x)\,g'(x)\,dx$$

$$\boxed{\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\cdot g(x)\,dx.}$$

$$\int u\cdot dv\,dx = u\cdot v - \int v\,du.$$

EX. $\int x\sin x\,dx$

let $u = x \quad dv = \sin x\,dx$

$du = dx \quad v = -\cos x$

$$-x\cos x + \int \cos x\,dx$$

$$-x\cos x - \sin x + C$$

- INTEGRATION BY PARTS, DEFINITE INTEGRALS.

$$\int_A^B u\,dv = uv\Big|_A^B - \int_A^B v\,du.$$

## § 5.7 ADDITIONAL TECHNIQUES OF INTEGRATION.

- Trig Substitution.

$\sqrt{a - x^2}$ $\quad x = a\sin\theta$

$\sqrt{a^2 + x^2}$ $\quad x = a\tan\theta$

$\sqrt{x^2 - a^2}$ $\quad x = a\sec\theta$

- Partial fractions (Appendix G)
- Integration using tables.

## Appendix G. PARTIAL FRACTIONS.

$f(x) = \dfrac{f(x)}{g(x)}$ } when both are polynomials.

PROPER RATIONAL FUNCTIONS.

degree of $f(x)$ < degree of $g(x)$.

IMPROPER RATIONAL FUNCTIONS.

degree of $g(x)$ > degree of $g(x)$

if we get an improper rational function we must first devide f(x) by g(x) until we have a remainder. until we have.

$$S(x) + \frac{R(x)}{g(x)}$$

where R(x) is the remainder of the devision. then we have a polynomial and a proper rational function.

the next step is to find the factors of g(x). then find the numerators for each root.

EX.
$$f(x) = \frac{x^3 + x}{x - 1} = S(x) + \frac{R(x)}{g(x)}$$

$$\begin{array}{r} x^2 + x + 2 \quad \leftarrow S(x) \\ x-1 \overline{) x^3 + 0 + x + 0} \\ \underline{x^3 - x^2} \\ x^2 + x \\ \underline{x^2 - x} \\ 2x + 0 \\ \underline{2x - 2} \\ 2 \quad \leftarrow R(x). \end{array}$$

EX.
$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}$$ b/c this is an improper rational function.

$$\int (x+1)\,dx + \int \frac{4x}{x^3 - x^2 - x + 1}$$

$$\frac{x^2}{2} + x + \int \frac{4x}{x^3 - x^2 - x + 1}$$

$\therefore$ we do partial fractions.

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{4x}{x^2(x-1) - (x-1)}$$

$$\frac{4x}{(x-1)(x^2-1)} = \frac{4x}{(x-1)(x+1)(x-1)}$$

$$\frac{4x}{(x-1)(x^2-1)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$$

CASE 2.

* KNOW DIFFERENT CASES.

$A = 2$
$B = 1$
$C = -1$

$\therefore$

$$\frac{x^2}{2} + x + \int \frac{2\,dx}{(x-1)^2} + \int \frac{dx}{(x-1)} - \int \frac{dx}{(x+1)}$$

$$\frac{x^2}{2} + x - \frac{2}{x-1} + C + \ln(x-1) - \ln(x+1)$$

APPENDIX G.

- long division
- factor $Q(x)$ - linear factors, irreducible Quadratics
- $\frac{R(x)}{Q(x)}$ = sum of partical fractions.
- Integrate parctal fractions.

$\frac{R(x)}{Q(x)}$ there are 4 cases (

CASE 1 All factors are linear.

When $Q(x) = (a_1x + b_1)(a_2x + b_2) \ldots (a_kx + b_k)$

then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{B}{a_2x + b_2} + \ldots + \frac{K}{a_kx + b_k}.$$

CASE 2 All factors are linear, some repeated.

When $Q(x) = (a_1x + b_1)(a_kx + b_k)^r$

then $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{B}{a_kx + b_k} + \frac{C}{(a_kx + b_k)^2} + \ldots$$

$$\ldots + \frac{K}{(a_kx + b_k)^r}$$

CASE 3 distinct linear, and irreducable quadratic factors.

When $Q(x) = (a_1x + b_1)(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0)$

Then $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{Bx^{n-1} + Cx^{n-2} + \ldots + Dx + E}{a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0}$$

CASE 4 repeated quadratic factors.

When $Q(x) = (a_1x + b_1)(a_2x^2 + b_2x + c)^r$

then $\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2x + B}{a_2x^2 + b_2x + c} + \dots + \frac{A_r + B}{(a_2x^2 + b_2x + c)^r}$

Theorm

if $(ax + b)^n$ is a factor of $Q(x)$
then $(ax + b)^{n-1}$ is also a factor.

note! we will be tested on CASE 1 & CASE 2.

§ 5.10 IMPROPER INTEGRALS

infinite integrals $\int_a^\infty f(x)\,dx$
discontinuous integrands $\int_a^b f(x)\,dx$

• INFINITE INTEGRALS

if $\int_a^t f(x)\,dx$ exists for every $t \geq a$
then $\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$

similarly...

$$\int_{-\infty}^b f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

finally...

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t \to \infty} \int_0^t f(x)\,dx + \lim_{t \to -\infty} \int_t^0 f(x)\,dx.$$

if $f(x)$ is continous on $[a, b)$ then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

and if between a & b there is a pt. of discont

c is discontinuous and $a < c < b$.

$$\int_a^b f(x)\,dx = \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

- CONVERGENT / DIVERGENT IMPROPER INTEGRALS

if the respective limit is an infinite number. then the improper integral is convergent; otherwise it is divergent.

EX. determine if $\int_0^3 \frac{dx}{x-1}$ is convergent; if yes evaluate the integral.

Solution

$$\int_0^3 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} + \lim_{t \to 1^+} \int_t^3 \frac{dx}{x-1}$$

$$= \lim_{t \to 1^-} \ln(x-1)\Big|_0^t + \lim_{t \to 1^+} \ln(x-1)\Big|_t^3$$

$$= \lim_{t \to 1^-} \ln(t-1) - \ln(-1) + \lim_{t \to 1^+} (t-1) - \ln(2)$$

b/c this can not be defined it is considered divergent. If it could be defined it would be considered convergent.

# CHAPTER 5 REVIEW

INDEFINITE INTEGRALS
- FUNCTION
- INVERSE OF DIFFERENTIATION.

DEFINITE INTEGRALS
- A NUMBER (NOT FUNCTION)
- PROPERTIES.

FUNDIMENTAL THEORM OF CALCULUS.

INTEGRATION TECHNIQUES
- SUBSTITUTION
- INTEGRATION BY PARTS
- PARTIAL FRACTIONS
- OTHERS
- TABLES

IMPROPER INTEGRALS
- INFINITE INTEGRALS
- DISCONTINUOUS INTEGRALS.
- CONVERGENT, DIVERGENT.

note $f(x) = \ln|x|$ $f'(x) = \frac{1}{x}$

$$\therefore \int \ln|x| dx = \frac{1}{x}$$

do not forget about absolute value.

# CHAPTER 6 APLICATIONS OF INTEGRATION.

* • geometric properties

* • average value of a function.

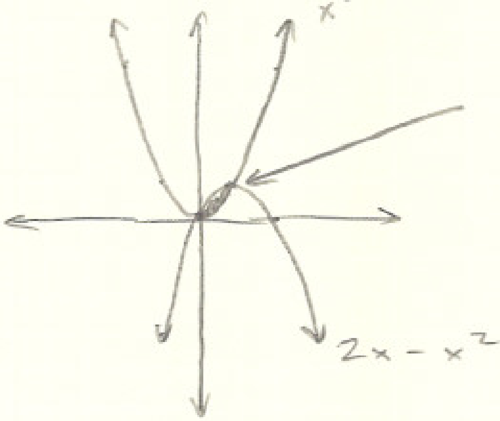
• physics and engineering aplications.

* • probability

• ODE (ordinary differential equations)
PDE (partical differential equations)

# § 6.1 AREAS.

EX.




FIND ENCLOSED AREA.



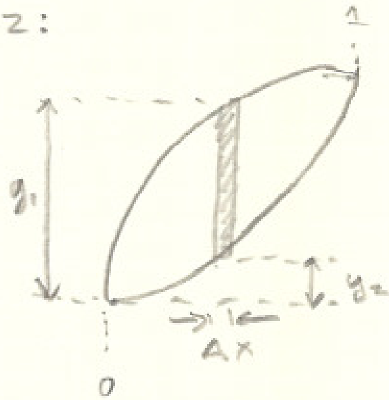
SOLUTION:

Step 1: find the intersecting points
set the equations equal.

$x^2 = 2x - x^2$

$x = 0 \ , \ x = 1$

Step 2:

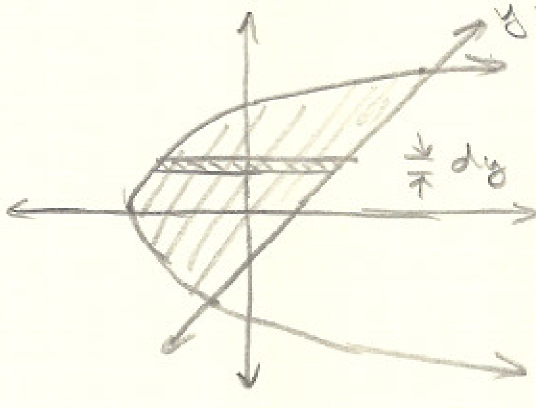



$$A = (y_1 - y_2)\,\Delta x$$
$$= \left((2x - x^2) - (x^2)\right)\Delta x$$
$$= (2x - 2x^2)\,\Delta x$$

$\therefore$ the area of the entire the entire thing can be expressed as

$$\int_0^1 (2x - 2x^2)\,dx = x^2 - \frac{2}{3}x^3 \Big|_0^1 = \frac{1}{3}$$

EX.




find the intesection points

$$\frac{y^2-6}{2} = y+1$$

$$y^2 - 6 = 2y + 2$$

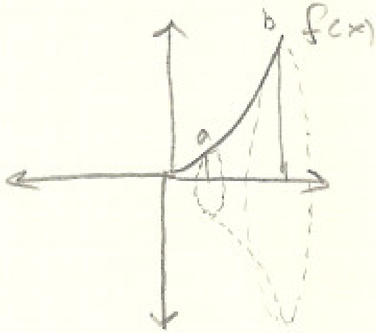
$$y^2 - 2y - 8 = 0 \qquad \therefore \quad y = +4 \quad y = -2$$

find the area of cross section.

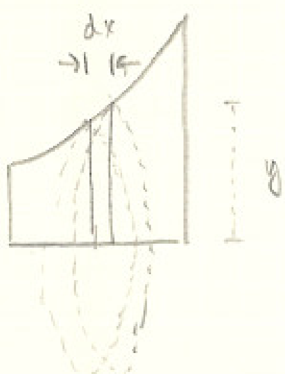
$$dA = (x_1 - x_2)\,\Delta y.$$
$$= \left[(x-1) + \left|\frac{y^2-6}{2}\right|\right]\Delta y$$

finished on page 445.

§ 6.2 VOLUMES.




FIND VOLUME OF f(x) FROM a to b ROTATED ABOUT THE x AXIS.

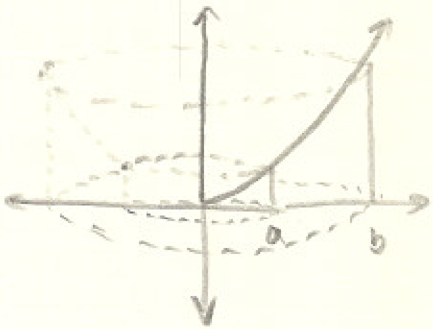


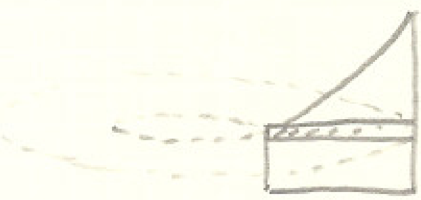
$$V = \int_a^b dV$$

$$dV = \pi y_1^2 \, dx$$

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

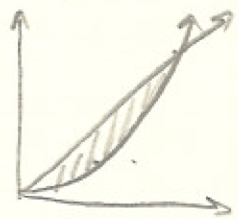
NOW SAME CURVE ROTATED ABOUT Y AXIS.




$$dV = \Big[\pi[g(b) - g(a)]^2 + \pi[g(b) - g(a)]^2$$

EX.

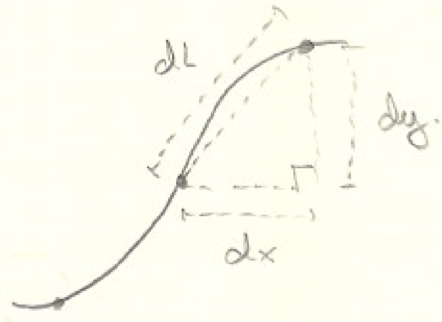


ROTATE ABOUT X AXIS.

$$dV = \pi [y_1^2 - y_2^2] \, dx$$

$$V = \pi \int [y_1^2 - y_2^2] \, dx$$

# §6.3 ARC LENGTH.




$$dL = \sqrt{(dx)^2 + (dy)^2}$$

$$dL = \sqrt{\frac{(dx)^2 + (dy)^2}{(dx)^2}}\, dx$$

$$dL = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$dL = \sqrt{1 + y'^2}\, dx$$

$$L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$$

or in the y direction

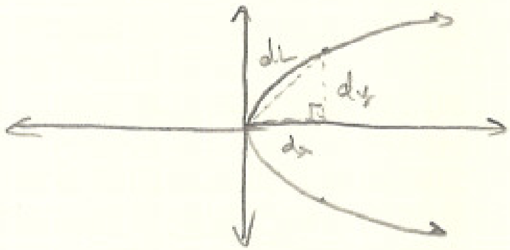
$$dL = \sqrt{dx^2 + dy^2}$$

$$dL = \sqrt{\frac{dx^2 + dy^2}{dy^2}}\, dy$$

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

$$L = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

EX. FIND THE LENGTH OF $y^2 = x$ from $(0,0)$ to $(1,1)$




SOLUTION IN TERMS OF x.

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

curve is $y^2 = x \quad \therefore \quad y = \sqrt{x}$

$$\therefore \quad y' = \frac{1}{2\sqrt{x}}$$

$$L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx$$

then solve integral.

SOLUTION IN TERMS OF y.

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

curve is $y^2 = x \quad \therefore \quad \frac{dx}{dy} = 2y$

$$L = \int_0^1 \sqrt{1 + 4y^2}\, dy.$$

now we need to make the integral fit the integration tables.

let $u = 2y$

$du = 2dy$

$dy = \frac{du}{2}$

$$L = \frac{1}{2}\int_0^2 \sqrt{1 + u^2}\, dy$$

using entry 31 from the table.

$$L = \frac{1}{2}\left[ \frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left| u + \sqrt{1+u^2} \right| \Bigg|_0^2 \right]$$

$$L = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(\sqrt{5}+2)$$

note: that in dealing with arc length, make sure to check both the x and y directions to see which is easier.

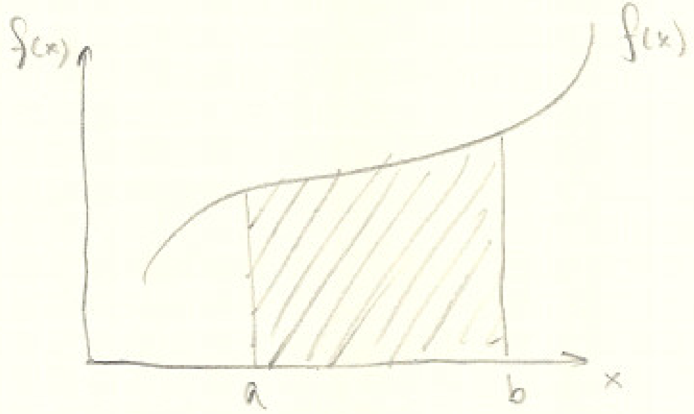
EX. $\displaystyle\int \frac{\cot x\, dx}{\sqrt{1+2\sin x}} = \int \frac{\cos x\, dx}{\sin x\sqrt{1+2\sin x}}$

Let $u = \sin x$

$du = \cos x\, dx$

$\displaystyle\int \frac{du}{u\sqrt{1+2u}}$ then use entry 49 from table.

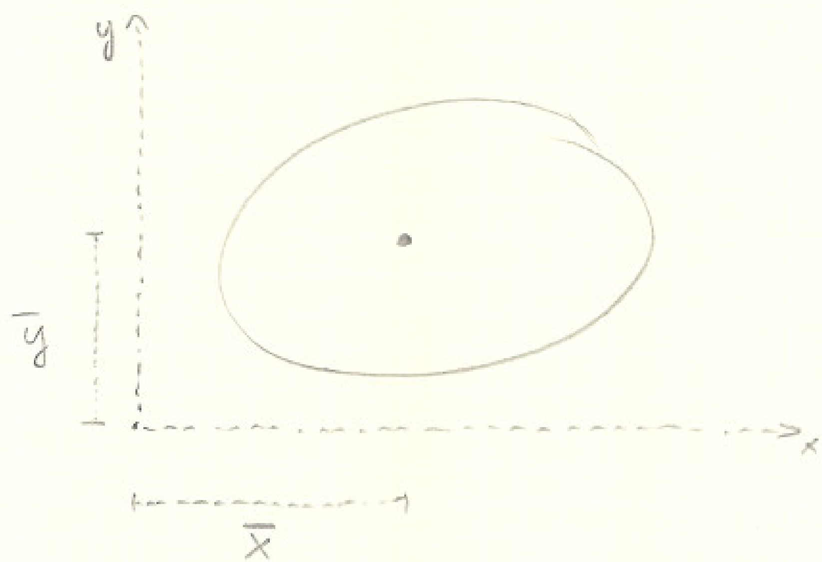
# §6.4 AVERAGE VALUE OF A FUNCTION.




$$A = \int_A^B f(x)\,dx$$

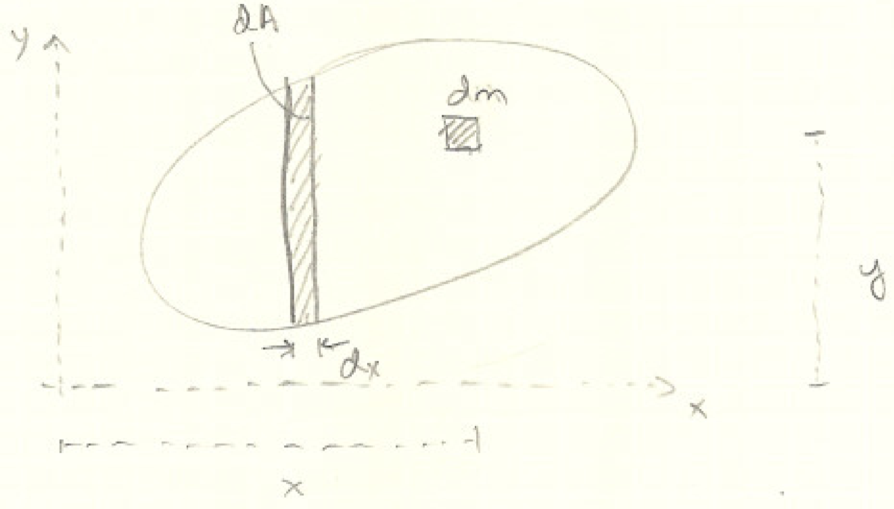
$$h = \frac{1}{B-A}\int_A^B f(x)\,dx = f(x) \text{ Avg.}$$

MASS CENTER, CENTER OF GRAVITY, CENTROID.




mass density, $\rho$ we will assume constant.

how to determine $\bar{x}$ and $\bar{y}$




$$dM_x = y\,dm$$

$$dM_y = x\,dm$$

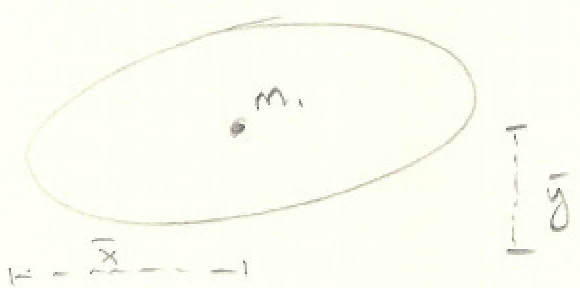
$$dm = \rho\,dA = \rho(y_1 - y_2)\,dx$$

$$dM_x = \frac{y_1 + y_2}{2}\,dm$$

$$dM_x = \rho\,\frac{y_1 + y_2}{2}(y_1 - y_2)\,dx$$

$$dM_x = \frac{\rho}{2}(y_1^2 - y_2^2)\,dx$$

$$M_x = \frac{\rho}{2}\int_A^B (y_1^2 - y_2^2)\,dx$$




$$M_x = \bar{y}\,m \qquad M_y = \bar{x}\,m$$

$$\bar{y} = \frac{M_x}{m}$$

$$\bar{y} = \frac{\frac{\rho}{2}\int_A^B (y_1^2 - y_2^2)\,dx}{\rho A} = \frac{1}{2A}\int_A^B (y_1^2 - y_2^2)\,dx$$

now we will find the moment about the y axis in order to find $\bar{x}$.

$$dM_y = x\,dm$$

$$dM_y = \rho x (y_1 - y_2)\,dx$$

$$M_y = \rho \int_A^B x (y_1 - y_2)\,dx$$

$$M_y = m\bar{x}$$

$$\bar{x} = \frac{\rho \int_A^B x(y_1 - y_2)\,dx}{\rho A} = \frac{1}{A}\int_A^B x(y_1 - y_2)\,dx$$

$$A = \int_A^B (y_2 - y_1)\,dx$$

Summary.

$$\bar{x} = \frac{\int_A^B x(y_1 - y_2)\,dx}{\int_A^B (y_1 - y_2)\,dx}$$

$$\bar{y} = \frac{\int_A^B (y_1^2 - y_2^2)\,dx}{2\int_A^B (y_1 - y_2)\,dx}$$

§ 6.7 PROBABILITY.

expectation and deviation

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx \qquad \sigma = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx$$

higher Moments

Normal distribution.

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

CHAPTER 6 REVIEW.

- $\left.\begin{array}{l}\text{differential}\\ \text{infinitesmal}\end{array}\right\}$ $dA, dV, dL, dm$
- upper and lower intervals
- integration

PARAMETRIC CURVES.

§1.7 PARAMETRIC CURVES
§3.5 THE CHAIN RULE p 225, 226.
§6.1 AREAS p445
§6.3 ARC LENGTH p463, 465